Recovery Guarantees of the Matrix Lasso Problem: On Equivalence of Convex and Non-Convex Formulations

Jinji Yang[1] **Xinyuan Song**[1] **Chungen Shen** [2] **Ziye Ma**[1*]
[1] Department of Computer Science, CityUHK [2] College of Science, USST
jinji.yang@cityu.edu.hk 00545027.city@cityu.edu.hk
shenchungen@usst.edu.cn ziyema@cityu.edu.hk
[*] Corresponding author

Abstract

Matrix sensing has long been studied as a pivotal low-rank optimization problem, important both for its diverse practical applications and as a theoretical tool for addressing non-convexity in neural network training. Traditional approaches to such non-convex problems typically fall into two categories: local search and convex relaxation. In this work, we examine a specific convex relaxation known as the matrix lasso, where the nuclear norm is used as a surrogate penalization for rank. Our key contribution is to show rigorously that this matrix lasso formulation can certifiably recover the ground truth matrix M^* with a low-rank solution of the same rank. We achieve this by establishing a new theoretical connection between the matrix lasso problem and its Burer–Monteiro factorized representation, a sophisticated approach inspired by results from the matrix completion literature. Furthermore, we extend our theoretical guarantees to scenarios involving finite-variance noise, underscoring the robustness of the matrix lasso method. Thus, our work fills a crucial gap in understanding the robustness of convex relaxations for matrix sensing, complementing existing results that primarily focus on non-convex factorized formulations.

1 Introduction

The low-rank matrix recovery problem (Recht et al., 2010; Candès and Plan, 2011; Mazumder et al., 2010; Negahban and Wainwright, 2011) has gained significant attention over the past two decades due to its wide range of applications in recommender systems, signal and image processing, control and system identification, and machine learning (see, e.g., Mazumder et al. (2010); Davenport and Romberg (2016); Fazel (2002); Zhou et al. (2015)). Mathematically, this problem seeks to recover a low-rank matrix $M \in \mathbb{R}^{n\times m}$ from noisy linear measurements given by $b = \mathcal{A}(M^*) + w$, where $\mathcal{A} : \mathbb{R}^{m\times n} \to \mathbb{R}^s$ is a linear measurement operator of the form $\mathcal{A}(M) = [\langle A_1, M\rangle, \dots, \langle A_s, M\rangle]^T$, $\{A_1, \dots, A_s\} \subseteq \mathbb{R}^{n\times m}$ are called sensing matrices, and w represents noise or other measurement errors. To search for a low-rank solution that best fits the observed vector, one natural approach is to consider the optimization problem Tao et al. (2022), which minimizes the objective function $\frac{1}{2}\|\mathcal{A}(M) - b\|^2$ subject to the rank constraint $\operatorname{rank}(M) \le r^*$, where $r^* = \operatorname{rank}(M^*)$. Zhang et al. (2021) prove that each asymmetric problem can be reformulated as an equivalent symmetric one:

$$\min_{M\in\mathbb{R}^{n\times n}} \frac{1}{2}\|\mathcal{A}(M) - b\|^2, \ \text{ s.t. } \operatorname{rank}(M) \le r^*, M \succeq 0, \tag{1}$$

where $M \succeq 0$ denotes that M is positive semidefinite. In this paper, we focus exclusively on the symmetric case. Since $\langle A, M\rangle = \langle \frac{A+A^\top}{2}, M\rangle$ holds for any symmetric matrix M, without loss of

Preprint. Under review.

generality, we may assume that the sensing matrices $A_i, i = 1, \ldots, s$ are symmetric. Although the model (1) guarantees that the solution exhibits low-rank structure, in many practical scenarios, the exact value of r^* is unknown. In this case, it is reasonable to consider the following regularized formulation Tao et al. (2022); Chen et al. (2020):

$$\min_{M \in \mathbb{R}^{n \times n}} \frac{1}{2} \|\mathcal{A}(M) - b\|^2 + \lambda \text{rank}(M), \text{ s.t. } M \succeq 0, \tag{2}$$

where $\lambda > 0$ is a regularization parameter. Theoretically, by appropriately tuning λ, it is possible to obtain a desirable low-rank solution. Since the rank function is non-convex and discontinuous, solving (2) is NP-hard in general, and it is impossible to compute a global optimal solution using an algorithm with polynomial-time complexity Tao et al. (2022). A common strategy is to replace $\text{rank}(M)$ with the nuclear norm $\|M\|_*$ as a convex surrogate, leading to the following matrix Lasso model Candès and Plan (2011); Wang et al. (2021); Negahban and Wainwright (2011):

$$\min_{M \in \mathbb{R}^{n \times n}} \frac{1}{2} \|\mathcal{A}(M) - b\|^2 + \lambda \|M\|_*, \text{ s.t. } M \succeq 0. \tag{3}$$

This model (3) serves as the main focus of this paper. Although (3) is a convex problem, its global minimizer M_{cvx} may not coincide with M^*. Therefore, it is important to establish guarantees on the maximum distance between any minimizer of (3) and M^*. While the nuclear norm is widely used to promote low-rank solutions, general theoretical proof of its effectiveness remains elusive, except in very specific cases. To address this challenge, inspired by the Burer–Monteiro approach Burer and Monteiro (2003), a non-convex method has been proposed Srebro et al. (2004). Specifically, $M \in \mathbb{R}^{m \times n}$ is factorized as $M = XY^T$ with low-rank factors $X \in \mathbb{R}^{m \times r}, Y \in \mathbb{R}^{n \times r}$. In the symmetric case, the factorization is simplified to $M = XX^T$. Motivated by the following result (Mazumder et al., 2010, Lemma 6),

$$\|M\|_* = \min_{XY^T = M,\ r \geq \text{rank}(M)} \frac{1}{2} (\|X\|_F^2 + \|Y\|_F^2), \tag{4}$$

we observed that if the solution M_{cvx} to (3) satisfies $\text{rank}(M_{\text{cvx}}) \leq r$, it must coincide with the solution to Chen et al. (2020); Mazumder et al. (2010)

$$\min_{X \in \mathbb{R}^{n \times r}} \frac{1}{2} \|\mathcal{A}(XX^T) - b\|^2 + \lambda \|X\|_F^2. \tag{5}$$

Although the model (5) naturally yields low-rank solutions and significantly reduces computational complexity, it still poses two major challenges. First, as in model (1), the choice of the rank parameter r remains an open issue. Second, model (5) is non-convex, making it difficult to guarantee finding a global optimal solution.

Nevertheless, despite potential differences between the solutions to (3) and (5) due to rank mismatches, the non-convex model (5) plays a crucial role in understanding the behavior of the convex relaxation (3). In this paper, we establish conditions under which the convex and non-convex formulations are equivalent, and further provide an error estimate for the convex model (3).

1.1 Related work

We briefly review existing literature on low-rank matrix recovery problems, with particular emphasis on the matrix Lasso model (3), which is the focus of our study.

We begin with defining the restricted isometry property. Recht et al. (2010) first proposed the extension of compressed sensing theory to the low-rank matrix recovery setting. In their context, the restricted isometry property (RIP), originally developed for sparse vector recovery, was generalized to matrices. This property plays a critical role in ensuring that sufficient information is retained.

Definition 1. *(Restricted Isometry Property) (Recht et al., 2010, Definition 3.1) Linear operator $\mathcal{A} : \mathbb{R}^{m \times n} \to \mathbb{R}^s$ satisfies (r, δ_r)-RIP if any $m \times n$ matrix M with $\text{rank} \leq r$,*

$$(1 - \delta_r)\|M\|_F^2 \leq \|\mathcal{A}(M)\|^2 \leq (1 + \delta_r)\|M\|_F^2. \tag{6}$$

1.1.1 Convex relaxation model

One of the first theoretical guarantees based on RIP for the matrix Lasso model (3) was first introduced by Candès and Plan (2011). They established error bounds for the matrix Dantzig selector and, by leveraging its connection to the matrix Lasso model (3), derived analogous estimation guarantees for the latter. Specifically, they showed that if the linear measurement operator $\mathcal{A}$ satisfies the $(4r^*, \delta)$-RIP with $\delta < (3\sqrt{2} - 1)/17$, and the noise w obeys $\|\mathcal{A}^*(w)\|_2 \leq \lambda/2$, then the solution M_{cvx} to the matrix Lasso problem satisfies $\|M_{\text{cvx}} - M^*\|_F \leq C_\delta \sqrt{r}\lambda$, where C_δ is a constant depending only on δ.

Further developments were made in Negahban and Wainwright (2011), where the authors analyzed the matrix Lasso model (3) and its variants under a restricted strong convexity condition. However, to the best of our knowledge, theoretical analysis of the convex model (3) has remained relatively limited.

A notable recent advance is attributed to Wang et al. (2021), which was designed to weaken the RIP requirement. They analyzed the recovery guarantees under the (tr^*, δ)-RIP with $t > 0$ and $\delta \leq \sqrt{(t-1)/t}$. The recovery conditions align with those in the constrained nuclear norm minimization setting discussed in Cai and Zhang (2014), ensuring exact recovery in the noiseless case.

Finally, McRae (2024) studied the conditions under which the matrix Lasso model has a unique low-rank solution. They showed that under the $(2r^*, \delta)$-RIP and the noise condition $\delta + \|\mathcal{A}^*(w)\|_2/\lambda \leq 1/16$, the solution to the matrix Lasso model is unique and satisfies $\operatorname{rank}(M_{\text{cvx}}) \leq 11r^*/10$.

Another widely studied convex approach to the low-rank matrix recovery problem is the constrained nuclear norm minimization model, typically formulated as

$$\min_{M \in \mathbb{R}^{m \times n}} \|M\|_*, \text{ s.t. } \mathcal{A}(M) - b \in \mathcal{S}, \tag{7}$$

where $\mathcal{S}$ is a bounded set. Recht et al. (2010) proved that exact recovery for the model (7) is possible in the noiseless case (i.e. $\mathcal{S} = \{0\}$) under suitable RIP assumption. Cai and Zhang (2013) established sharp thresholds, showing that exact recovery is guaranteed if the operator satisfies $\delta_{r^*} < 1/3$. Cai and Zhang (2014) extended the analysis to the more general (tr^*, δ)-RIP condition with $t > 4/3$, and provided guarantees for both noiseless and noisy settings. More recently, Yalcin et al. (2023) investigated recovery performance in the special case where $M \succeq 0$ and provided conditions under which exact recovery is guaranteed via the SDP model.

1.1.2 Non-convex model

Due to the fact that the Burer–Monteiro factorization inherently enforces the low-rank structure, recent non-convex approaches often directly tackle the following formulation:

$$\min_{X \in \mathbb{R}^{n \times r}} \frac{1}{2}\|\mathcal{A}(XX^T) - b\|^2. \tag{8}$$

This line of research primarily focuses on analyzing the optimization landscape of the non-convex formulation (8), with particular emphasis on identifying RIP conditions that guarantee the absence of spurious local minima (i.e., local minima that are not global) in the noiseless setting, as well as on establishing RIP-based error bounds for local minimizers in the presence of noise.

In the noiseless setting, a series of works have established increasingly sharp RIP thresholds to ensure the absence of spurious local minima in non-convex matrix sensing. Bhojanapalli et al. (2016) showed that $\delta \leq 1/5$ suffices for exact recovery, while subsequent work by Zhang et al. (2019); Zhang and Zhang (2020) proved that the bound $\delta < 1/2$ is tight. This condition was further extended to general objectives by Bi and Lavaei (2020).

In the presence of noise, Zhang et al. (2018) demonstrated that $\delta \leq 1/35$ ensures that all local minima are near the ground truth. Ma et al. (2022) sharpened this result for the quadratic objective, showing that $\delta < 1/2$ is both necessary and sufficient, even under general finite-variance noise. More recently, Ma and Sojoudi (2023) extended these results to general objectives.

Most of the aforementioned results are established under the assumption that the factorization rank exactly matches the true rank, i.e., $r = r^*$. In contrast, over-parameterization (i.e., using a rank $r > r^*$) may also arise in practical situations and has recently attracted increasing attention. Ma et al. (2023) established global error bounds and polynomial-time convergence under arbitrary initialization.

Lastly, unlike the regularization (4) derived from nuclear norm, asymmetric matrix sensing often incorporates regularization $\|X^TX - Y^TY\|_F^2$ to align asymmetric factorizations with symmetric models Ge et al. (2017); Zhang et al. (2021).

Table 1: Comparison of low-rank matrix recovery models

Model	Convex/ Non-convex	Parameter Dependence	Advantages	Disadvantages
(2)	Non-convex	λ	Direct rank control; interpretable	NP-hard; no poly-time global solver
(3)	Convex	λ	Global optimum via convex solvers;	$\mathrm{rank}(M_{\mathrm{cvx}})$ may not march r^*
(5)	Non-convex	r, λ	Low-rank solution; Low storage and computational cost	Must choose r; possible spurious local minima
(7)	Convex	$\mathcal{S}$	Exact recovery in noiseless case; no rank tuning	Large-scale SDP hard
(8)	Non-convex	r	Low-rank solution; Low storage and computational cost	Requires correct r; spurious local minima

1.2 Motivation: Insights from Matrix Completion

The matrix completion problem can be viewed as a special case of the matrix sensing problem, where the observations take the form $b = \mathcal{P}_\Omega(M^*) + w$, where Ω denotes the set of observed entries, and $\mathcal{P}_\Omega$ is the projection onto the set Ω. In general, the matrix completion problem does not satisfy the RIP condition. Chen et al. (2020) studied the following convex regularized formulation:

$$\min_{M \in \mathbb{R}^{m\times n}} \frac{1}{2}\|\mathcal{P}_\Omega(M) - b\|^2 + \lambda\|M\|_*. \tag{9}$$

They further analyzed its non-convex counterpart

$$\min_{X\in\mathbb{R}^{m\times r},\, Y\in\mathbb{R}^{n\times r}} \frac{1}{2}\|\mathcal{P}_\Omega(XY^\top) - b\|^2 + \frac{\lambda}{2}(\|X\|_F^2 + \|Y\|_F^2), \tag{10}$$

and proved that there exists an approximate critical point of this non-convex formulation (10) serving as an extremely tight approximation to the solution of the convex problem. This equivalence allows the statistical guarantees of the non-convex model (10) to be transferred to its convex model (9). Our work is inspired by this line of analysis in Chen et al. (2020).

Many existing analyses of matrix sensing, such as that of Wang et al. (2021), do not fully exploit first-order information. These studies typically derive error bounds that apply to any point whose objective function value lies below that of the ground truth M^*, without requiring the point to be a stationary point or local minimizer. In contrast, our analysis adopts the strategy that is closely aligned with the approach in Chen et al. (2020), which explicitly utilizes the first-order optimality conditions of stationary points to derive sharper error estimation guarantees.

1.3 Our contributions

This paper makes the following key contributions:

1. We establish a novel RIP-based bound for the composition $\mathcal{A}^*\mathcal{A}$ in the low-rank setting. Specifically, in Proposition 1, we show that if the linear operator $\mathcal{A}$ satisfies the (r, δ_r)-RIP with $\delta_r < 2/3$, then for any matrix M of rank at most r, the operator $\mathcal{A}^*\mathcal{A}$ is well-conditioned in the Frobenius norm.

2. Building on Proposition 1, we demonstrate that under appropriate conditions, the solution to the convex formulation (3) coincides with a stationary point of the non-convex problem (5).

3. We establish sufficient conditions under which the non-convex objective exhibits local strong convexity in a neighborhood of the ground truth. This improves the result in Chen et al. (2020), which only guarantees restricted strong convexity. Building on this local curvature property, we further prove the existence of a stationary point near the ground truth.

4. Our main theoretical result establishes sufficient conditions for the convex formulation (3) to have a unique solution whose rank exactly matches that of the ground truth matrix. We also derive an error bound. To the best of our knowledge, this is the first result that rigorously guarantees rank recovery for the convex model (3) in the noisy setting.

Table 2: Comparison of theoretical recovery guarantees for the matrix Lasso model (3).

Reference	**Assumptions**	**Error estimation**	$\mathrm{rank}(M_{\mathrm{cvx}})$/ **uniqueness**
Candès and Plan (2011)	$(4r^*, \delta)$-RIP; $\delta < (3\sqrt{2}-1)/17$; $\|\mathcal{A}^*(w)\|_2 \leq \lambda/2$	$C_\delta \sqrt{r^*}\,\lambda$	– / –
Wang et al. (2021)	(tr^*, δ)-RIP; $\delta \leq \sqrt{(t-1)/t}$; $\|w\| \leq \varepsilon$	$C_{\lambda/\varepsilon,\delta}\sqrt{r^*}\lambda$	– / –
McRae (2024)	$(2r^*, \delta)$-RIP; $\delta + \|\mathcal{A}^*(w)\|_2/\lambda \leq 1/16$	–	$\leq 1.1\, r^*$ / Yes
This work	(11) and (12)	(13)	r^* / Yes

Table 2 summarizes the theoretical recovery guarantees for the matrix Lasso model (3). Our analysis achieves several notable improvements. First, while the RIP requirement in our work is relatively stringent compared to Wang et al. (2021), the derived error estimation (13) is significantly sharper. Second, unlike prior works such as Candès and Plan (2011) and Wang et al. (2021), which do not explicitly ensure the uniqueness or the exact rank of the recovered solution, our result guarantees that the convex solution exactly recovers the rank-r^* structure and is unique. This contributes to a better understanding of the rank recovery behavior of the matrix Lasso model.

Notations Denote by $\mathcal{A}^*$ the adjoint operator of $\mathcal{A}$, so that $\mathcal{A}^*(x)$ is a symmetric matrix for any $x \in \mathbb{R}^s$. Let $M^* \in \mathbb{R}^{n\times n}$ be the ground truth matrix with rank r^* and $M^* \succeq 0$, we denote by $X^* \in \mathbb{R}^{n\times r^*}$ a factor of M^* such that $M^* = X^* X^{*T}$. $\|v\|$ denotes the Euclidean norm of a vector v, while $\|M\|_F$ and $\|M\|_2$ denote the Frobenius norm and the operator norm (or induced ℓ_2 norm) of a matrix M, respectively. Denote by $\mathcal{O}^n$ the set of $n \times n$ orthonormal matrices. For a matrix $X \in \mathbb{R}^{n\times r}$ with full column rank, $\sigma_i(X)$ denotes its i-th largest singular value and $\sigma_{\min}(X)$ denotes its smallest nonzero singular value, (i.e., the r-th largest singular value). κ denotes the condition number of the ground truth factor X^*, i.e. $\kappa = \|X^*\|_2/\sigma_{\min}(X^*)$. For a point $X \in \mathbb{R}^{n\times r}$ and a radius $\mathcal{R} > 0$, we define the closed ball centered at X with radius $\mathcal{R}$ as $\mathcal{B}(X, \mathcal{R}) := \{Y \in \mathbb{R}^{n\times r} \mid \|Y - X\|_F \leq \mathcal{R}\}$.

2 Main results

We first present our main results. In contrast to the main result in Chen et al. (2020), our analysis leverages the properties of stationary points, which allows us to conclude not only that the solution to the convex model (3) is unique, but also that $\mathrm{rank}(M_{\mathrm{cvx}}) = \mathrm{rank}(M^*)$. These results are formally established in the following two theorems, which cover both the noiseless and noisy scenarios.

Theorem 1. *Suppose that the linear operator $\mathcal{A}$ satisfies the $(2r^*, \delta)$-RIP condition, the observation is noiseless (i.e., $w = 0$), and the RIP constant δ satisfies $2\sqrt{r^*}\kappa^2 \leq \sqrt{(2-\delta)/(8\delta - 2\delta^2)}$. Then the solution M_{cvx} to the convex model* (3) *is unique, with* $\mathrm{rank}(M_{\mathrm{cvx}}) = r^*$. *Moreover, for any $\varepsilon > 0$, there exists $\lambda_0 > 0$ such that for all $0 < \lambda \leq \lambda_0$, we have $\|M_{\mathrm{cvx}} - M^*\|_F \leq \varepsilon$.*

We remark that Theorem 1 follows as a direct corollary by setting $w = 0$ in Theorem 2.

Theorem 2. *Suppose that the linear operator $\mathcal{A}$ satisfies the $(2r^*, \delta)$-RIP condition, and the noise intensity $\|\mathcal{A}^*(w)\|_2$ and the RIP constant δ satisfy*

$$\frac{(1-\delta)^2\|X^*\|_2^2}{4\sqrt{r^*}\kappa^4(3-\delta)} > \|\mathcal{A}^*(w)\|_2 + \lambda, \quad \|\mathcal{A}^*(w)\|_2 < (1-\theta)\lambda. \tag{11}$$

for some constant $\theta \in (0, 1)$. Then if the RIP constant δ satisfies

$$\frac{\phi^2}{(1-\delta)^2\|X^*\|_2^2} + \frac{2\phi}{(1-\delta)} < \theta\lambda\sqrt{\frac{2-\delta}{8\delta - 2\delta^2}}, \tag{12}$$

with $\phi = \sqrt{r^*}\kappa^2(\|\mathcal{A}^*(w)\|_2 + \lambda)$, the solution M_{cvx} to convex model (3) *is unique, with* $\mathrm{rank}(M_{\mathrm{cvx}}) = r^*$*, and the following error bound holds:*

$$\|M_{\mathrm{cvx}} - M^*\|_F \leq \frac{\phi^2}{(1-\delta)^2\|X^*\|_2^2} + \frac{2\phi}{(1-\delta)}. \tag{13}$$

Remark 1. *For condition* (12) *, once the regularization parameter* λ *satisfies* $\lambda > \|\mathcal{A}^*(w)\|_2$*, a suitable* $\theta \in (0,1)$ *can always be chosen. For condition* (12)*, the left-hand side decreases as* δ *becomes smaller, while the right-hand side increases to infinity as* $\delta \to 0$ *due to the term* $\sqrt{(2-\delta)/(8\delta - 2\delta^2)}$*. Consequently, there always exists* $\delta \in (0,1)$ *such that the inequality is satisfied. It is also worth noting that when the problem is more challenging, specifically when the target rank* r^**, the condition number* κ*, or the noise intensity* $\|\mathcal{A}^*(w)\|_2$ *are large, or* $\|X^*\|_2$ *is small, the left side of* (12) *becomes larger. In such scenarios, a smaller RIP constant* δ *is required to ensure that condition* (12) *remains feasible.*

Remark 2. *From condition* (12)*, error estimation* (13) *and the expression of* ϕ*, it is intuitive that* λ *should be chosen close to* $\|\mathcal{A}^*(w)\|_2$*. However, the actual value must also satisfy condition* (12)*, which imposes an additional dependency on the RIP constant* δ*. Therefore,* λ *must be carefully selected to balance both conditions.*

Compared to the results in Wang et al. (2021), our recovery bound (13) offers improved tightness. Most importantly, our results assert that the recovered matrix is of the same rank of M^*, which is a new discovery thanks to our approach of linking convex and non-convex formulations. Specifically, (Wang et al., 2021, Theorem 4) provides the following error bound for their convex model:

$$\|M_{\mathrm{cvx}} - M^*\|_F \leq \frac{\sqrt{r^*}\beta_1(5+2\beta_2)\lambda + 2(1+4\beta_2+2\beta_2^2)\varepsilon}{\sqrt{r^*}(1-\beta_2)\lambda}(\sqrt{r^*}\beta_1\lambda + \varepsilon), \tag{14}$$

where $\beta_1 = 2/((1-\delta)\sqrt{1+\delta})$ and $\beta_2 = \delta/(\sqrt{(1-\delta^2)(t-1)})$. To illustrate the superiority of our estimation, we consider the following numerical example:

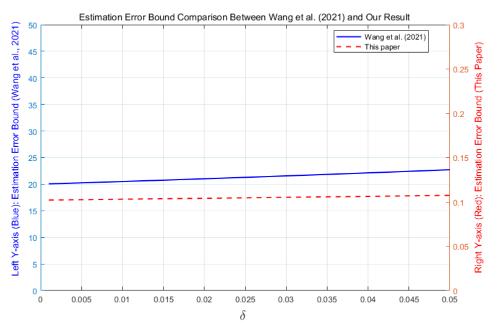

Figure 1: Comparison of the estimation error bounds obtained in this paper and in Wang et al. (2021). Assume that $\mathcal{A}$ satisfies the $(2,\delta)$-RIP condition, $\mathrm{rank}(M^*) = 1$, and $\|M^*\|_2 = 100$. The noise level is set as $w = 0.001$. The regularization parameter λ used in Wang et al. (2021) is chosen according to (Wang et al., 2021, (20)), while, in our estimation, we set $\lambda = 0.05$.

3 On Equivalence of Convex and Non-convex Formulations

This section outlines the proof strategy and introduces the key technical tools supporting our main result. We begin by establishing a connection between the convex formulation (3) and its non-convex counterpart (5). In particular, we investigate the optimization landscape of the non-convex objective

$$f(X) = \frac{1}{2}\|\mathcal{A}(XX^T) - b\|^2 + \lambda\|X\|_F^2.$$

Subsequently, in the next section, we show that, under suitable conditions, this non-convex objective admits a critical point sufficiently close to the ground truth. This key result enables us to connect the solution of the convex model (3) to the ground truth.

Our derivation strategy is partly inspired by the two-stage analytical framework in Chen et al. (2020), where the authors study matrix completion by first establishing a connection between the convex and non-convex formulations, and then proving-via a leave-one-out approach combined with mathematical induction-that the non-convex objective admits a point with small gradient norm close to the ground truth. In contrast, we study matrix sensing under the restricted isometry property, where linear operator $\mathcal{A}$ replace the projections used in matrix completion. Moreover, our error bounds are given explicitly in terms of model parameters, avoiding undetermined or implicit constants. This ensures that all intermediate results are precise and computable, enhancing both interpretability and practical relevance.

We present a fundamental property of the RIP, which plays a crucial role in our analysis. The following proposition shows that under the RIP condition, the norm of $\mathcal{A}^*(\mathcal{A}(M))$ can be bounded in terms of $\|M\|_F$. The proof is provided in Appendix A.1.

Proposition 1. *Suppose the linear operator $\mathcal{A}: \mathbb{R}^{m\times n} \to \mathbb{R}^s$ satisfies the (r, δ_r)-RIP with $r \geq 2, \delta_r < \frac{2}{3}$ and let $\mathcal{A}^*$ denote its adjoint. Then for any M with $\operatorname{rank}(M) \leq r$, it holds that*

$$\frac{2-3\delta_r}{2+\delta_r}\|M\|_F^2 \leq \|\mathcal{A}^*\mathcal{A}M\|_F^2 \leq \frac{2+3\delta_r}{2-\delta_r}\|M\|_F^2.$$

From Proposition 1, it immediately follows that

$$\begin{aligned} \|\mathcal{A}^*\mathcal{A}M - M\|_F^2 &= \|\mathcal{A}^*\mathcal{A}M\|_F^2 + \|M\|_F^2 - 2\langle \mathcal{A}^*\mathcal{A}M, M\rangle \\ &\leq \tfrac{2+3\delta_r}{2-\delta_r}\|M\|_F^2 + \|M\|_F^2 - 2(1-\delta_r)\|M\|_F^2 = \tfrac{8\delta_r - 2\delta_r^2}{2-\delta_r}\|M\|_F^2. \end{aligned} \tag{15}$$

We also recall some properties of the nuclear norm, particularly its subdifferential structure. Let $M \in \mathbb{R}^{n\times n}$ be a rank-r positive semidefinite matrix with eigenvalue decomposition $M = Q\Sigma Q^T$, where $Q \in \mathbb{R}^{n\times r}$ has orthonormal columns and $\Sigma \in \mathbb{R}^{r\times r}$ is diagonal with nonnegative entries. The tangent space at M, denoted by T, is defined as (see Chen et al. (2020)): $T = \{QA^T + BQ^T \mid A, B \in \mathbb{R}^{n\times r}\}$. Let $\mathcal{P}_T(\cdot)$ denote the orthogonal projection onto the subspace T. Then for any $X \in \mathbb{R}^{n\times n}$, the projection of X onto T is given by Chen et al. (2020) (see also (Candès and Recht, 2009, (3.5)))

$$\mathcal{P}_T(X) = QQ^TX + XQQ^T - QQ^TXQQ^T. \tag{16}$$

Let $T^\perp$ denote the orthogonal complement of T, and $\mathcal{P}_{T^\perp}(\cdot)$ the corresponding orthogonal projection. Then the subdifferential of the nuclear norm at M is characterized by (see (Candès and Recht, 2009, (3.4)); see also (Cai et al., 2010, (2.6)))

$$\partial\|M\|_* = \{QQ^T + W \mid W \in T^\perp,\ \|W\|_2 \leq 1\}. \tag{17}$$

The following lemma implies that the projection of the gradient $\nabla f(X)$ onto the orthogonal complement space $T^\perp$ can be controlled by the regularization parameter, provided that XX^T is sufficiently close to M^*. A corresponding result appears as Claim 2 in Chen et al. (2020), where a similar bound is established in the context of matrix completion. The proof is provided in Appendix A.2.

Lemma 1. *Suppose that the ground truth M^* with $\operatorname{rank}(M^*) = r^*$ and the linear operator $\mathcal{A}$ satisfies the (r, δ)-RIP. If there exists X with $\operatorname{rank}(X) = r^*$, parameters $\alpha, \beta, \gamma > 0$ such that:*

$$\|XX^T - M^*\|_F < \alpha\lambda\sqrt{\frac{2-\delta}{8\delta - 2\delta^2}},\ \ \|\mathcal{A}^*(w)\|_2 < \beta\lambda,\ \ \frac{\|\nabla f(X)\|_F}{\sigma_{\min}(X)} < \gamma\lambda,\ \ \alpha+\beta+\gamma < 1.$$

Then, it holds that

$$\|\mathcal{P}_T(S)\|_F \leq \frac{\|\nabla f(X)\|_F}{\sigma_{\min}(X)}, \quad \|\mathcal{P}_{T^\perp}(S)\|_2 \leq (\alpha+\beta+\gamma)\lambda,$$

where

$$S = -\lambda QQ^T - \mathcal{A}^*(\mathcal{A}(XX^T) - b), \tag{18}$$

$X = Q\Lambda P^T$ is singular value decomposition of X with $Q \in \mathbb{R}^{n\times r^*}, \Lambda \in \mathbb{R}^{r^*\times r^*}, P \in \mathbb{R}^{r^*\times r^*}$ and T is the tangent space of XX^T.

The following theorem establishes a connection between the convex model (3) and the non-convex model (5). Specifically, if there exists a point such that $\nabla f(X)$ is small and XX^T is close to M^*, then the solution to model (3) is guaranteed to be close to XX^T as well. The proof is provided in Appendix A.3.

Theorem 3. *Suppose that the ground truth $\operatorname{rank}(M^*) = r^*$ and the linear operator $\mathcal{A}$ satisfies the $(2r^*, \delta)$-RIP. Assume that there exists X with $\operatorname{rank}(X) = r^*$, parameters $\alpha, \beta, \gamma, \tau > 0$ such that:*

$$\|XX^T - M^*\|_F < \alpha\lambda\sqrt{\frac{2-\delta}{8\delta - 2\delta^2}},\ \ \|\mathcal{A}^*(w)\|_2 < \beta\lambda,\ \ \frac{\|\nabla f(X)\|_F}{\sigma_{\min}(X)} < \gamma\lambda,\ \ \alpha+\beta+\gamma+\tau = 1.$$

We further assume that the gradient of the objective function $f(X)$ in model (5) *satisfies:*

$$\|\nabla f(X)\|_F \leq \tau\lambda\sigma_{\min}(X)\sqrt{\frac{r^*(1-\delta)}{n(1+\delta)}}. \tag{19}$$

Then, for any solution $M_{\rm cvx}$ to model (3)*,*

$$\|M_{\rm cvx} - XX^T\|_F \leq \frac{32\|\nabla f(X)\|_F}{(1-\delta)\sigma_{\min}(X)}.$$

4 The stationary point near the ground truth

In this section, we establish the existence of a stationary point in the neighborhood of the ground truth that satisfies the assumptions required in the Theorem 3. We begin by proving that the function f exhibits local smoothness and local strong convexity in a neighborhood of the ground truth X^*. The proof of Lemma 2 is provided in Appendix A.4.

Lemma 2. *Suppose that the linear operator $\mathcal{A}$ satisfies the $(2r^*, \delta)$-RIP condition and $\|\mathcal{A}^*(w)\|_2 + \lambda \leq 3(1+\delta)C\|X^*\|_2^2$ with $C = \frac{\xi(1-\sigma)(1-\delta)}{\kappa^2(6-2\delta)}$ where the parameters $\xi \in (0,1], \sigma \in (0,1)$. Then, the function f is L_s-smooth over the closed ball $\mathcal{B}(X^*, C\|X^*\|_2)$ with $L_s = 10(1+\delta)\|X^*\|_2^2$.*

We note that the condition: $\|\mathcal{A}^*(w)\|_2 + \lambda \leq 3(1+\delta)C\|X^*\|_2^2$ is not strictly necessary. It is imposed primarily for the sake of simplifying the expression of L_s without introducing additional assumptions in the following analysis. Moreover, the parameters ξ and σ are chosen to ensure that the neighborhood (ball) considered here is consistent with that in the subsequent lemma, which facilitates the argument in the following lemmas and theorems.

Due to the presence of the RIP property in the matrix sensing problem, as opposed to the matrix completion setting considered in Chen et al. (2020), where only a restricted local strong convexity was established, we can derive the strong convexity of $f(X)$ in a neighborhood around X^*, as shown in the following lemma. Its proof is provided in Appendix A.5.

Lemma 3. *Suppose that the linear operator $\mathcal{A}$ satisfies the $(2r^*, \delta)$-RIP condition and $\|\mathcal{A}^*(w)\|_2 - \lambda \leq (3-\delta)C\|X^*\|_2^2$ with $C = \frac{\xi(1-\sigma)(1-\delta)}{\kappa^2(6-2\delta)}$, where the parameters $\xi \in (0,1], \sigma \in (0,1)$. Then, the function f is L_c-strong convex over the closed ball $\mathcal{B}(X^*, C\|X^*\|_2)$ with $L_c = 4\sigma(1-\delta)\sigma_{\min}^2(X^*)$.*

Now, we proceed to show that a stationary point of $f(X)$ can be found close to the ground truth M^*. Specifically, we apply the gradient descent method (Algorithm 1) to the model (5) to locate this stationary point. Furthermore, we use mathematical induction to prove that the generated sequence remains in the closed ball $\mathcal{B}(X^*, C\|X^*\|_2)$ throughout the iterations. This technique has also been used in Chen et al. (2020) to control the iterations within a predefined neighborhood, providing a useful framework for our proof. The proof of Theorem 4 is provided in Appendix A.6.

Algorithm 1: Construction of an approximate solution.

Initialization: $X^0 = X^*$.

Gradient descent: for $k = 0, 1, \ldots$ do

$$X^{k+1} = X^k - \eta\nabla f(X^k) = X^k - \eta(\mathcal{A}^*\mathcal{A}(X^k {X^k}^T - M)X^k + \lambda X^k),$$

here $\eta > 0$ is the step size.

Theorem 4. *Suppose that the linear operator $\mathcal{A}$ satisfies the $(2r^*, \delta)$-RIP condition, and*

$$\frac{(1-\delta)^2\|X^*\|_2^2}{4\kappa^4(3-\delta)} > \sqrt{r^*}(\|\mathcal{A}^*(w)\|_2 + \lambda). \tag{20}$$

Let the sequences X^k be generated by Algorithm 1, and suppose that the step size in Algorithm 1 satisfies

$$\eta \leq \frac{L_c - \frac{2\sqrt{r^*}(\|\mathcal{A}^*(w)\|_2 + \lambda)}{C}}{L_s^2 - (\frac{2\sqrt{r^*}(\|\mathcal{A}^*(w)\|_2+\lambda)}{C})^2}, \tag{21}$$

where $C = \frac{\xi(1-\sigma)(1-\delta)}{\kappa^2(6-2\delta)}$, $\xi \in (\frac{4\sqrt{r^*}\kappa^4(3-\delta)(\|\mathcal{A}^*(w)\|_2+\lambda)}{(1-\delta)^2\|X^*\|_2^2}, 1]$, *and* $\sigma = \frac{1}{2}$. *Then, for any* $X^k \in \{X^k\}$,

$$\|X^k H^k - X^*\|_F \le C\|X^*\|_2,$$

where $H^k := \arg\min_{R \in \mathcal{O}^{r^*}} \|X^k R - X^*\|_F$.

Next, we prove that with a proper choice of the step size, the sequence of function values exhibits a sufficient decrease. The proof is provided in Appendix A.7.

Lemma 4. *Under the assumptions of Theorem 4, if the step size also satisfies*

$$\eta \le \min\left\{ \frac{L_c - \frac{2\sqrt{r^*}(\|\mathcal{A}^*(w)\|_2+\lambda)}{C}}{L_s^2 - (\frac{2\sqrt{r^*}(\|\mathcal{A}^*(w)\|_2+\lambda)}{C})^2}, \frac{1}{L_s} \right\},$$

then

$$f(X^{k+1}) \le f(X^k) - \frac{\eta}{2}\|\nabla f(X^k)\|_F^2.$$

Based on the result of Lemma 4, we now show that the sequence generated by Algorithm 1 has all accumulation points which are stationary points of the non-convex model (5). The proof is provided in Appendix A.8.

Lemma 5. *Assume that the same conditions as in Theorem 4 hold. Then, every accumulation point of* $\{X^k H^k\}$ *lies in* $\mathcal{B}(X^*, C\|X^*\|_2)$ *and is a stationary point of the non-convex formulation* (5)*.*

Based on the above preparations, we now present an error estimation for the non-convex model (5). The proof is provided in Appendix A.9.

Theorem 5. *Suppose that the linear* $\mathcal{A}$ *satisfies the* $(2r^*, \delta)$*-RIP condition and*

$$\frac{(1-\delta)^2\|X^*\|_2^2}{4\kappa^4(3-\delta)} > \sqrt{r^*}(\|\mathcal{A}^*(w)\|_2 + \lambda).$$

Let $\bar{M} = \bar{X}\bar{X}^T$, *where* $\bar{X}$ *is an accumulation point of* $\{X^k H^k\}$ *and* $H^k = \arg\min_{R \in \mathcal{O}^{r^*}} \|X^k R - X^*\|_F$.
Then, it holds that

$$\|\bar{M} - M^*\|_F \le \frac{r^*(\|\mathcal{A}^*(w)\|_2 + \lambda)^2\kappa^4}{(1-\delta)^2\|X^*\|_2^2} + 2\frac{\sqrt{r^*}(\|\mathcal{A}^*(w)\|_2 + \lambda)\kappa^2}{(1-\delta)}.$$

5 Proof of main result

By combining Theorems 3 and 5, we establish the proof of Theorem 2.

Proof of Theorem 2 . From Theorem 5, the nonconvex problem admits a stationary point $\bar{X}$ such that $\nabla f(\bar{X}) = 0$. This ensures that the condition (19) in Theorem 3 is satisfied. Letting $\alpha \to \theta$, $\beta \to 1-\theta, \gamma \to 0, \tau \to 0$, Theorem 3 yields

$$\|M_{\mathrm{cvx}} - \bar{M}\|_F = 0$$

for any local solution M_{cvx} to the convex model (3). Here, $\bar{M}$ is the matrix obtained in Theorem 5. This implies that M_{cvx} is unique, $\mathrm{rank}(M_{\mathrm{cvx}}) = \mathrm{rank}(\bar{M}) = r^*$, and the bound (13) holds. □

6 Conclusion

To summarize, this work advances the theoretical understanding of low-rank matrix recovery via convex and non-convex approaches in matrix sensing. We introduce a new RIP-based bound for $\mathcal{A}^*\mathcal{A}$, generalize the convex–nonconvex equivalence to matrix sensing, and provide new insights into the landscape of the non-convex formulation near the ground truth. Most notably, we offer the first rigorous result ensuring that the convex solution not only achieves low recovery error but also exactly matches the rank of the ground truth under suitable conditions.

References

Beck, A. (2017). *First-order methods in optimization*. SIAM.

Bhojanapalli, S., Kyrillidis, A., and Sanghavi, S. (2016). Dropping convexity for faster semi-definite optimization. In *Conference on Learning Theory*, pages 530–582. PMLR.

Bi, Y. and Lavaei, J. (2020). Global and local analyses of nonlinear low-rank matrix recovery problems. arXiv:2010.04349.

Burer, S. and Monteiro, R. D. (2003). A nonlinear programming algorithm for solving semidefinite programs via low-rank factorization. *Mathematical Programming*, 95(2):329–357.

Cai, J.-F., Candès, E. J., and Shen, Z. (2010). A singular value thresholding algorithm for matrix completion. *SIAM Journal on Optimization*, 20(4):1956–1982.

Cai, T. T. and Zhang, A. (2013). Sharp RIP bound for sparse signal and low-rank matrix recovery. *Applied and Computational Harmonic Analysis*, 35(1):74–93.

Cai, T. T. and Zhang, A. (2014). Sparse representation of a polytope and recovery of sparse signals and low-rank matrices. *IEEE Transactions on Information Theory*, 60(1):122–132.

Candès, E. J. (2008). The restricted isometry property and its implications for compressed sensing. *Comptes Rendus Mathematique*, 346(9):589–592.

Candès, E. J. and Plan, Y. (2011). Tight oracle inequalities for low-rank matrix recovery from a minimal number of noisy random measurements. *IEEE Transactions on Information Theory*, 57(4):2342–2359.

Candès, E. J. and Recht, B. (2009). Exact matrix completion via convex optimization. *Foundations of Computational Mathematics*, 9(6):717–772.

Chen, X., Chi, Y., Fan, J., Ma, C., and Yan, Y. (2020). Noisy matrix completion: Understanding statistical guarantees for convex relaxation via nonconvex optimization. *SIAM Journal on Optimization*, 30(4):3098–3121.

Davenport, M. A. and Romberg, J. (2016). An overview of low-rank matrix recovery from incomplete observations. *IEEE Journal of Selected Topics in Signal Processing*, 10(4):608–622.

Fazel, M. (2002). *Matrix rank minimization with applications*. PhD thesis, Stanford University.

Ge, R., Jin, C., and Zheng, Y. (2017). No spurious local minima in nonconvex low rank problems: A unified geometric analysis. In *Proceedings of the 34th International Conference on Machine Learning*, volume 70 of *Proceedings of Machine Learning Research*, pages 1233–1242.

Ma, Z., Bi, Y., Lavaei, J., and Sojoudi, S. (2022). Sharp restricted isometry property bounds for low-rank matrix recovery problems with corrupted measurements. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 36, pages 7672–7681.

Ma, Z., Bi, Y., Lavaei, J., and Sojoudi, S. (2023). Geometric analysis of noisy low-rank matrix recovery in the exact parametrized and the overparametrized regimes. *INFORMS Journal on Optimization*.

Ma, Z. and Sojoudi, S. (2023). Noisy low-rank matrix optimization: Geometry of local minima and convergence rate. In *International Conference on Artificial Intelligence and Statistics*, pages 3125–3150. PMLR.

Mazumder, R., Hastie, T., and Tibshirani, R. (2010). Spectral regularization algorithms for learning large incomplete matrices. *Journal of Machine Learning Research*, 11:2287–2322.

McRae, A. D. (2024). Low solution rank of the matrix lasso under RIP with consequences for rank-constrained algorithms. *arXiv preprint arxiv.org/abs/2404.12828*.

Negahban, S. and Wainwright, M. J. (2011). Estimation of (near) low-rank matrices with noise and high-dimensional scaling. *Annals of Statistics*, 39(2):1069–1097.

Recht, B., Fazel, M., and Parrilo, P. A. (2010). Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization. *SIAM Review*, 52(3):471–501.

Srebro, N., Rennie, J., and Jaakkola, T. (2004). Maximum-margin matrix factorization. *Advances in neural information processing systems*, 17.

Tao, T., Qian, Y., and Pan, S. (2022). Column $\ell_{2,0}$-norm regularized factorization model of low-rank matrix recovery and its computation. *SIAM Journal on Optimization*, 32(2):959–988.

Wang, W., Zhang, F., and Wang, J. (2021). Low-rank matrix recovery via regularized nuclear norm minimization. *Applied and Computational Harmonic Analysis*, 54:1–19.

Yalcin, B., Ma, Z., Lavaei, J., and Sojoudi, S. (2023). Semidefinite programming versus Burer-Monteiro factorization for matrix sensing. In *Proceedings of the AAAI Conference on Artificial Intelligence*.

Zhang, G. and Zhang, R. Y. (2020). How many samples is a good initial point worth in low-rank matrix recovery? In *Advances in Neural Information Processing Systems*, volume 33, pages 12583–12592.

Zhang, H., Bi, Y., and Lavaei, J. (2021). General low-rank matrix optimization: Geometric analysis and sharper bounds. In *Advances in Neural Information Processing Systems*, volume 33, pages 27369–27380.

Zhang, R. Y., Josz, C., Sojoudi, S., and Lavaei, J. (2018). How much restricted isometry is needed in nonconvex matrix recovery? In *Advances in Neural Information Processing Systems*, volume 31.

Zhang, R. Y., Sojoudi, S., and Lavaei, J. (2019). Sharp restricted isometry bounds for the inexistence of spurious local minima in nonconvex matrix recovery. *Journal of Machine Learning Research*, 20(114):1–34.

Zhou, X., Yang, C., Zhao, H. X., and Yu, C. (2015). Low-rank modeling and its applications in image analysis. *ACM Computing Surveys*, 47:1–33.

A Technical Appendices and Supplementary Material

A.1 Proof of Proposition 1

To facilitate our analysis, we first recall Lemma 2.1 in Candès (2008), which establishes a fundamental property of the RIP condition and will be used in our proof.

Lemma 6. *(Candès, 2008, Lemma 2.1) Suppose the linear operator $\mathcal{A} : \mathbb{R}^{m\times n} \to \mathbb{R}^s$ satisfies the $(2r, \delta_{2r})$-RIP. Then, for any M_1, M_2 with $\operatorname{rank}(M_1),\ \operatorname{rank}(M_2) \le r$, it holds that*

$$|\langle \mathcal{A}(M_1), \mathcal{A}(M_2)\rangle - \langle M_1, M_2\rangle| \le \frac{\delta_{2r}}{2}\left(\|M_1\|_F^2 + \|M_2\|_F^2\right).$$

Proof of Proposition 1. Without causing confusion, we simplify the expressions as follows: we denote the vector $\mathcal{A}M := \mathcal{A}(M)$, the matrix $\mathcal{A}^*\mathcal{A}M := \mathcal{A}^*(\mathcal{A}(M))$, and the vector $\mathcal{A}\mathcal{A}^*\mathcal{A}M := \mathcal{A}(\mathcal{A}^*(\mathcal{A}(M)))$. For each i, define the matrix $M_i \in \mathbb{R}^{m\times n}$ such that its k-th row satisfies $[M_i]_{k*} = [M]_{k*}$ for $k = i$ and $[M_i]_{k*} = \vec{0}$ for $k \ne i$, where $[M]_{k*}$ denote the k-th row of M, $\vec{0}$ is the $1\times n$ zero vector. Then, $M = \sum_i M_i$. Since $\mathcal{A}$ is linear, we have

$$\begin{aligned}
& |\langle \mathcal{A}\mathcal{A}^*\mathcal{A}M, \mathcal{A}M\rangle - \langle \mathcal{A}^*\mathcal{A}M, M\rangle| \\
= & |\langle \mathcal{A}(\textstyle\sum_i \mathcal{A}^*\mathcal{A}M_i), \mathcal{A}(\sum_i M_i)\rangle - \langle \sum_i \mathcal{A}^*\mathcal{A}M_i, \sum_i M_i\rangle| \\
= & |\langle \textstyle\sum_i \mathcal{A}(\mathcal{A}^*\mathcal{A}M_i), \sum_i \mathcal{A}(M_i)\rangle - \langle \sum_i \mathcal{A}^*\mathcal{A}M_i, \sum_i M_i\rangle| \\
= & |\textstyle\sum_i \langle \mathcal{A}(\mathcal{A}^*\mathcal{A}M_i), \mathcal{A}(M_i)\rangle - \langle \mathcal{A}^*\mathcal{A}M_i, M_i\rangle| \\
\le & \textstyle\sum_i |\langle \mathcal{A}(\mathcal{A}^*\mathcal{A}M_i), \mathcal{A}(M_i)\rangle - \langle \mathcal{A}^*\mathcal{A}M_i, M_i\rangle|.
\end{aligned}$$

Since $\operatorname{rank}(\mathcal{A}^*\mathcal{A}M_i) \le 1 \le \frac{r}{2}$, applying Lemma 6 yields

$$\begin{aligned}
& |\langle \mathcal{A}\mathcal{A}^*\mathcal{A}M, \mathcal{A}M\rangle - \langle \mathcal{A}^*\mathcal{A}M, M\rangle| \\
\le & \textstyle\sum_i \frac{\delta_r}{2}(\|\mathcal{A}^*\mathcal{A}M_i\|_F^2 + \|M_i\|_F^2) \\
= & \tfrac{\delta_r}{2}(\|\mathcal{A}^*\mathcal{A}M\|_F^2 + \|M\|_F^2).
\end{aligned}$$

Thus,

$$-\frac{\delta_r}{2}(\|\mathcal{A}^*\mathcal{A}M\|_F^2 + \|M\|_F^2) \le \|\mathcal{A}^*\mathcal{A}M\|_F^2 - \|\mathcal{A}M\|^2 \le \frac{\delta_r}{2}(\|\mathcal{A}^*\mathcal{A}M\|_F^2 + \|M\|_F^2). \tag{22}$$

From the left-hand side of (22), we have

$$\|\mathcal{A}M\|^2 - \frac{\delta_r}{2}(\|\mathcal{A}^*\mathcal{A}M\|_F^2 + \|M\|_F^2) \le \|\mathcal{A}^*\mathcal{A}M\|_F^2.$$

Using the definition of the RIP,

$$\begin{aligned}
(1-\delta_r)\|M\|_F^2 - \tfrac{\delta_r}{2}\|M\|_F^2 &\le \|\mathcal{A}^*\mathcal{A}M\|_F^2 + \tfrac{\delta_r}{2}\|\mathcal{A}^*\mathcal{A}M\|_F^2, \\
\tfrac{2-3\delta_r}{2+\delta_r}\|M\|_F^2 &\le \|\mathcal{A}^*\mathcal{A}M\|_F^2.
\end{aligned}$$

Similarly, from the right-hand side of (22), we obtain

$$\|\mathcal{A}^*\mathcal{A}M\|_F^2 \le \frac{2+3\delta_r}{2-\delta_r}\|M\|_F^2.$$

which concludes the proof. □

A.2 Proof of Lemma 1

Proof of Lemma 1. From the definition of $f(X)$, we obtain

$$2\mathcal{A}^*(\mathcal{A}(XX^T) - b)X = -2\lambda X + \nabla f(X). \tag{23}$$

By right-multiplying both sides of (18) with X and substituting the result into the left-hand side of (23), we have

$$-2\lambda QQ^T X - 2SX = -2\lambda X + \nabla f(X).$$

Substituting $X = Q\Lambda P^T$ into the previous equation, we get

$$-\lambda QQ^\top Q\Lambda P^T - SQ\Lambda P^T = -\lambda Q\Lambda P^T + \frac{1}{2}\nabla f(X),$$

which leads to

$$SQ = -\frac{1}{2}\nabla f(X)P\Lambda^{-1}.$$

Hence,

$$\|SQ\|_F \leq \frac{1}{2}\|\nabla f(X)P\|_F\|\Lambda^{-1}\|_2 \leq \frac{1}{2}\|\nabla f(X)\|_F\|P\|_2\|\Lambda^{-1}\|_2 = \frac{\|\nabla f(X)\|_F}{2\sigma_{\min}(X)}.$$

Thus, using the projection formulation (16), (see (Chen et al., 2020, (65))) we estimate

$$\begin{aligned}\|\mathcal{P}_T(S)\|_F \leq& \|QQ^TS + SQQ^T - QQ^TSQQ^T\|_F \\ =& \|QQ^TS(I - QQ^T) + SQQ^T\|_F \\ \leq& \|Q^TS(I - QQ^T)\|_F + \|SQ\|_F \\ \leq& 2\|SQ\|_F \leq \frac{\|\nabla f(X)\|_F}{\sigma_{\min}(X)}.\end{aligned}$$

We now prove $\|P_{T^\perp}(S)\|_2 \leq (\alpha + \beta + \gamma)\lambda$. From (18) and $b = \mathcal{A}(M^*) + w$, we have

$$S + \lambda QQ^T + XX^T = M^* - \mathcal{A}^*\mathcal{A}(XX^T - M^*) + \mathcal{A}^*(w) + XX^T - M^*.$$

Substituting $X = Q\Lambda P^T$ into the left-hand side of the previous equation:

$$S + \lambda QQ^T + Q\Lambda^2Q^T = M^* + (XX^T - M^*) - \mathcal{A}^*\mathcal{A}(XX^T - M^*) + \mathcal{A}^*(w).$$

Due to $S = \mathcal{P}_{T^\perp}(S) + \mathcal{P}_T(S)$, we obtain

$$\mathcal{P}_{T^\perp}(S) + \lambda QQ^T + Q\Lambda^2Q^T = M^* + (XX^T - M^*) - \mathcal{A}^*\mathcal{A}(XX^T - M^*) + \mathcal{A}^*(w) - \mathcal{P}_T(S).$$

Appling Weyl's inequality, we have for $i > r^*$,

$$\begin{aligned}&\sigma_i(M^* + (XX^T - M^*) - \mathcal{A}^*\mathcal{A}(XX^T - M^*) + \mathcal{A}^*(w) - \mathcal{P}_T(S)) \\ \leq&\sigma_i(M^*) + \|(XX^T - M^*) - \mathcal{A}^*\mathcal{A}(XX^T - M^*) + \mathcal{A}^*(w) - \mathcal{P}_T(S)\|_2 \\ \leq&\|(XX^T - M^*) - \mathcal{A}^*\mathcal{A}(XX^T - M^*)\|_2 + \|\mathcal{A}^*(w)\|_2 + \|\mathcal{P}_T(S)\|_2 \\ \leq&\|(XX^T - M^*) - \mathcal{A}^*\mathcal{A}(XX^T - M^*)\|_F + \|\mathcal{A}^*(w)\|_2 + \|\mathcal{P}_T(S)\|_F \\ <&(\alpha + \beta + \gamma)\lambda < \lambda,\end{aligned}$$

where the second inequality uses the fact that $\operatorname{rank}(M^*) = r^*$, and the fourth inequality follows from (15) and conditions on α, β, γ. On the other hand, for $i \leq r^*$,

$$\sigma_i\left(\lambda QQ^T + Q\Lambda^2Q^T\right) \geq \lambda.$$

Since $\lambda QQ^T + Q\Lambda^2Q^T \in T$, it follows that

$$\|\mathcal{P}_{T^\perp}(S)\|_2 < (\alpha + \beta + \gamma)\lambda,$$

which concludes the proof. □

A.3 Proof of Theorem 3

Proof of Theorem 3. Let $\Delta = M_{\text{cvx}} - XX^T$. By the definition of M_{cvx}, we consider the following inequality for Δ:

$$\lambda\|XX^T\|_* + \frac{1}{2}\|\mathcal{A}(XX^T) - b\|^2 \geq \lambda\|XX^T + \Delta\|_* + \frac{1}{2}\|\mathcal{A}(XX^T + \Delta) - b\|^2.$$

Using the convexity of the nuclear norm $\|\cdot\|_*$ and its subgradient $QQ^T + W$, where $Q \in \mathbb{R}^{n\times r^*}$, of XX^T given in (17), the previous inequality leads to

$$-\lambda\langle QQ^T, \Delta\rangle - \lambda\langle W, \Delta\rangle - \left\langle\mathcal{A}(XX^T) - b, \mathcal{A}(\Delta)\right\rangle \geq \frac{1}{2}\|\mathcal{A}(\Delta)\|^2.$$

Choosing W such that $\langle W, \Delta \rangle = \|\mathcal{P}_{T^\perp}(\Delta)\|_*$ (Candès and Recht, 2009, Lemma 3.1)(also see (Chen et al., 2020, Lemma 6)), we have

$$-\lambda \langle QQ^T, \Delta \rangle - \lambda \|\mathcal{P}_{T^\perp}(\Delta)\|_* - \left\langle \mathcal{A}^*(\mathcal{A}(XX^T) - b), \Delta \right\rangle \geq \frac{1}{2}\|\mathcal{A}(\Delta)\|^2.$$

Substituting the expression of S from (18), we obtain

$$\langle S, \Delta \rangle - \lambda \|\mathcal{P}_{T^\perp}(\Delta)\|_* = \langle \mathcal{P}_T(S), \Delta \rangle + \langle \mathcal{P}_{T^\perp}(S), \Delta \rangle - \lambda \|\mathcal{P}_{T^\perp}(\Delta)\|_* \geq \frac{1}{2}\|\mathcal{A}(\Delta)\|^2 \geq 0. \quad (24)$$

Applying the Cauchy-Schwarz inequality and duality inequality to (24) leads to

$$-(\|\mathcal{P}_T(S)\|_F \|\mathcal{P}_T(\Delta)\|_F + \|\mathcal{P}_{T^\perp}(S)\|_2 \|\mathcal{P}_{T^\perp}(\Delta)\|_*) + \lambda \|\mathcal{P}_{T^\perp}(\Delta)\|_* \leq 0.$$

Using the bound on $\|\mathcal{P}_{T^\perp}(S)\|_2$ from Lemma 1, we find that

$$\|\mathcal{P}_T(S)\|_F \|\mathcal{P}_T(\Delta)\|_F \geq (\lambda - \|\mathcal{P}_{T^\perp}(S)\|_2) \|\mathcal{P}_{T^\perp}(\Delta)\|_* \geq \tau\lambda \|\mathcal{P}_{T^\perp}(\Delta)\|_*.$$

Moreover, by the upper bound on $\|\mathcal{P}_T(S)\|_F$, we further have

$$\frac{\|\nabla f(X)\|_F}{\sigma_{\min}(X)} \|\mathcal{P}_T(\Delta)\|_F \geq \tau\lambda \|\mathcal{P}_{T^\perp}(\Delta)\|_*. \quad (25)$$

By the gradient condition (19) and $\sqrt{\frac{r^*(1-\delta)}{n(1+\delta)}} \leq 1$, it follows that

$$\|\mathcal{P}_T(\Delta)\|_F \geq \|\mathcal{P}_{T^\perp}(\Delta)\|_* \geq \|\mathcal{P}_{T^\perp}(\Delta)\|_F. \quad (26)$$

Next, from (24), we can also derive

$$\begin{aligned}
\frac{1}{2}\|\mathcal{A}(\Delta)\|^2 &\leq \|\mathcal{P}_T(S)\|_F \|\mathcal{P}_T(\Delta)\|_F - (\lambda - \|\mathcal{P}_{T^\perp}(S)\|_2) \|\mathcal{P}_{T^\perp}(\Delta)\|_* \\
&\leq \|\mathcal{P}_T(S)\|_F \|\mathcal{P}_T(\Delta)\|_F - \tau\lambda \|\mathcal{P}_{T^\perp}(\Delta)\|_* \\
&\leq \|\mathcal{P}_T(S)\|_F \|\mathcal{P}_T(\Delta)\|_F \\
&\leq \frac{\|\nabla f(X)\|_F \|\Delta\|_F}{\sigma_{\min}(X)},
\end{aligned} \quad (27)$$

where the last inequality uses the upper bound on $\|\mathcal{P}_T(S)\|_F$ and $\|\mathcal{P}_T(\Delta)\|_F \leq \|\Delta\|_F$. Additionally, using the RIP condition of the linear operator $\mathcal{A}$, we have

$$\begin{aligned}
\|\mathcal{A}(\Delta)\| = \|\mathcal{A}(\mathcal{P}_T(\Delta)) + \mathcal{A}(\mathcal{P}_{T^\perp}(\Delta))\| &\geq \|\mathcal{A}(\mathcal{P}_T(\Delta))\| - \|\mathcal{A}(\mathcal{P}_{T^\perp}(\Delta))\| \\
&\geq \sqrt{1-\delta}\|\mathcal{P}_T(\Delta)\|_F - \sqrt{\frac{n}{r^*}(1+\delta)} \|\mathcal{P}_{T^\perp}(\Delta)\|_F.
\end{aligned} \quad (28)$$

In the last inequality, we use that $\operatorname{rank}(\mathcal{P}_T(\Delta)) \leq 2r^*$ and $\operatorname{rank}(\mathcal{P}_{T^\perp}(\Delta)) \leq n - 2r^*$. Furthermore, the matrix $\mathcal{P}_{T^\perp}$ can be decomposed into at most $\lceil \frac{n-2r^*}{r^*} \rceil$ components, each being a matrix of rank at most r^*. Substituting the gradient condition (19) into (25), we obtain

$$\sqrt{\frac{n}{r^*}(1+\delta)} \|\mathcal{P}_{T^\perp}(\Delta)\|_F \leq \sqrt{\frac{n}{r^*}(1+\delta)} \|\mathcal{P}_{T^\perp}(\Delta)\|_* \leq \frac{\sqrt{1-\delta}}{2} \|\mathcal{P}_T(\Delta)\|_F,$$

which substitutes into (28) implies

$$\|\mathcal{A}(\Delta)\| \geq \frac{\sqrt{1-\delta}}{2} \|\mathcal{P}_T(\Delta)\|_F.$$

Finally, from (26), we obtain

$$\|\Delta\|_F \leq \|\mathcal{P}_T(\Delta)\|_F + \|\mathcal{P}_{T^\perp}(\Delta)\|_F \leq 2\|\mathcal{P}_T(\Delta)\| \leq \frac{4}{\sqrt{1-\delta}} \|\mathcal{A}(\Delta)\|.$$

Substituting it into (27), we conclude

$$\|\Delta\|_F \leq \frac{32\|\nabla f(X)\|_F}{(1-\delta)\sigma_{\min}(X)}.$$

□

A.4 Proof of Lemma 2

Proof of Lemma 2. For any $V \in \mathbb{R}^{n\times r}$, we have

$$\begin{aligned}
&\langle V,\ \nabla^2 f(X)[V]\rangle \\
=\ & 2\langle V,\ \mathcal{A}^*(\mathcal{A}(XV^T+VX^T))X+\mathcal{A}^*(\mathcal{A}(XX^T)-b)V\rangle+2\lambda\|V\|_F^2 \\
=\ & 4\langle VX^T,\ \mathcal{A}^*(\mathcal{A}(VX^T))\rangle+2\langle VV^T,\ \mathcal{A}^*(\mathcal{A}(XX^T-X^*X^{*T}))\rangle \\
& -2\langle VV^T,\ \mathcal{A}^*(w)\rangle+2\lambda\|V\|_F^2 \\
=\ & 4\|\mathcal{A}(VX^T)\|^2+2\langle\mathcal{A}(VV^T),\ \mathcal{A}(XX^T-X^*X^{*T})\rangle-2\langle VV^T,\ \mathcal{A}^*(w)\rangle \\
& +2\lambda\|V\|_F^2,
\end{aligned} \tag{29}$$

where the second equality uses the fact that $\mathcal{A}(XV^T)=\mathcal{A}(VX^T)$. Applying the Cauchy-Schwarz inequality and duality inequality, we obtain

$$\begin{aligned}
&\langle V,\ \nabla^2 f(X)[V]\rangle \\
\le\ & 4\|\mathcal{A}(VX^T)\|^2+2\|\mathcal{A}(VV^T)\|\|\mathcal{A}(XX^T-X^*X^{*T})\|+2\|VV^T\|_*\|\mathcal{A}^*(w)\|_2 \\
& +2\lambda\|V\|_F^2.
\end{aligned} \tag{30}$$

For the first term, invoking the RIP property, we have

$$\begin{aligned}
4\|\mathcal{A}(VX^T)\|^2 \le\ & 4(1+\delta)\|VX^T\|_F^2 \\
=\ & 4(1+\delta)\langle V^TV, X^TX\rangle \\
\le\ & 4(1+\delta)\|V^TV\|_*\|X^TX\|_2 \\
\le\ & 4(1+\delta)\|V\|_F^2(\|XX^T-X^*X^{*T}\|_2+\|X^*X^{*T}\|_2),
\end{aligned} \tag{31}$$

where the last inequality follows from $\|V^TV\|_*=\|V\|_F^2$ and the triangle inequality. Similarly, for the second term, applying the RIP property yields

$$2\|\mathcal{A}(VV^T)\|\|\mathcal{A}(XX^T-X^*X^{*T})\|\le 2(1+\delta)\|VV^T\|_F\|XX^T-X^*X^{*T}\|_F. \tag{32}$$

Substituting (31) and (32) into (30), we obtain

$$\begin{aligned}
&\langle V,\ \nabla^2 f(X)[V]\rangle \\
\le\ & 4(1+\delta)\|V\|_F^2\|X^*\|_2^2+6(1+\delta)\|V\|_F^2\|XX^T-X^*X^{*T}\|_F+2\|V\|_F^2\|\mathcal{A}^*(w)\|_2 \\
& +2\lambda\|V\|_F^2.
\end{aligned} \tag{33}$$

Observe that $\|XX^T-X^*X^{*T}\|_F$ can be bounded as

$$\begin{aligned}
&\|XX^T-X^*X^{*T}\|_F \\
=\ & \|XX^T-XX^{*T}+XX^{*T}-X^*X^{*T}\|_F \\
\le\ & \|XX^T-XX^{*T}\|_F+\|XX^{*T}-X^*X^{*T}\|_F \\
\le\ & \|X\|_2\|X^T-X^{*T}\|_F+\|X-X^*\|_F\|X^{*T}\|_2 \\
\le\ & (\|X-X^*\|_2+\|X^*\|_2)\|X-X^*\|_F+\|X-X^*\|_F\|X^*\|_2 \\
\le\ & \|X-X^*\|_F^2+2\|X^*\|_2\|X-X^*\|_F.
\end{aligned} \tag{34}$$

Substituting (34) into (33) yields

$$\begin{aligned}
&\langle V,\ \nabla^2 f(X)[V]\rangle \\
\le\ & (4(1+\delta)\|X^*\|_2^2+6(1+\delta)(\|X-X^*\|_F^2+2\|X^*\|_2\|X-X^*\|_F)+2\|\mathcal{A}^*(w)\|_2+2\lambda)\|V\|_F^2.
\end{aligned}$$

When $X\in\mathcal{B}(X^*, C\|X^*\|_2)$, it follows that

$$\begin{aligned}
&\langle V,\ \nabla^2 f(X)[V]\rangle \\
\le\ & [4(1+\delta)\|X^*\|_2^2+6(1+\delta)(C^2+2C)\|X^*\|_2^2+2\|\mathcal{A}^*(w)\|_2+2\lambda]\|V\|_F^2.
\end{aligned}$$

Furthermore, under the assumption $\|\mathcal{A}^*(w)\|_2+\lambda\le 3(1+\delta)C\|X^*\|_2^2$, we have

$$\langle V,\ \nabla^2 f(X)[V]\rangle\le[4(1+\delta)\|X^*\|_2^2+6(1+\delta)(C^2+3C)\|X^*\|_2^2]\|V\|_F^2.$$

Finally, noting that $C=\frac{\xi(1-\sigma)(1-\delta)}{\kappa^2(6-2\delta)}\le\frac{1}{4}$, we obtain

$$\langle V,\ \nabla^2 f(X)[V]\rangle\le 10(1+\delta)\|X^*\|_2^2\|V\|_F^2,$$

which implies that f is L_s- smooth with $L_s=10(1+\delta)\|X^*\|_2^2$. $\square$

A.5 Proof of Lemma 3

Proof of Lemma 3. Applying the Cauchy-Schwarz inequality and duality inequality to (29), we obtain

$$\begin{array}{rl} & \langle V,\ \nabla^2 f(X)[V]\rangle \\ \geq & 4\|\mathcal{A}(VX^T)\|^2 - 2\|\mathcal{A}(VV^T)\|\|\mathcal{A}(XX^T - X^*X^{*T})\| - 2\|VV^T\|_*\|\mathcal{A}^*(w)\|_2 \\ & +2\lambda\|V\|_F^2. \end{array} \tag{35}$$

Applying the RIP property to the first term, we have

$$\begin{array}{rl} 4\|\mathcal{A}(VX^T)\|^2 \geq & 4(1-\delta)\|XV^T\|_F^2 \\ = & 4(1-\delta)\langle V^TV, X^TX\rangle \\ = & 4(1-\delta)\langle V^TV, X^{*T}X^* - X^{*T}X^* + X^TX\rangle \\ \geq & 4(1-\delta)(\|VX^{*T}\|_F^2 - \|V^TV\|_F\|X^TX - X^{*T}X^*\|_F). \end{array} \tag{36}$$

Substituting (36) and (32) into (35) yields

$$\begin{array}{rl} & \langle V,\ \nabla^2 f(X)[V]\rangle \\ \geq & 4(1-\delta)\|VX^{*T}\|_F^2 - 4(1-\delta)\|V^TV\|_F\|X^TX - X^{*T}X^*\|_F \\ & -2(1+\delta)\|VV^T\|_F\|XX^T - X^*X^{*T}\|_F - 2\|V\|_F^2\|\mathcal{A}^*(w)\|_2 + 2\lambda\|V\|_F^2. \end{array} \tag{37}$$

Similar to (34), we have the following bound:

$$\|X^TX - X^{*T}X^*\|_F \leq \|X - X^*\|_F^2 + 2\|X^*\|_2\|X - X^*\|_F. \tag{38}$$

Putting (38) and (34) into (37), we obtain

$$\begin{array}{rl} & \langle V,\ \nabla^2 f(X)[V]\rangle \\ \geq & 4(1-\delta)\|VX^{*T}\|_F^2 - (6-2\delta)\|VV^T\|_F(\|X - X^*\|_F^2 + 2\|X^*\|_2\|X - X^*\|_F) \\ & -\|V\|_F^2\|\mathcal{A}^*(w)\|_2 + 2\lambda\|V\|_F^2 \\ \geq & 4(1-\delta)\|V\|_F^2\sigma_{\min}^2(X^*) - (6-2\delta)\|V\|_F^2(\|X - X^*\|_F^2 + 2\|X^*\|_2\|X - X^*\|_F) \\ & -\|V\|_F^2\|\mathcal{A}^*(w)\|_2 + 2\lambda\|V\|_F^2. \end{array}$$

Due to $X \in \mathcal{B}(X^*, C\|X^*\|_2)$ and the assumption $-\|\mathcal{A}^*(w)\|_2 + \lambda \geq -(3-\delta)C\|X^*\|_2^2$, it follows that

$$\langle V,\ \nabla^2 f(X)[V]\rangle \geq (4(1-\delta)\sigma_{\min}^2(X^*) - (6-2\delta)(C^2+3C)\|X^*\|_2^2)\|V\|_F^2.$$

By the definition of $C = \frac{\xi(1-\sigma)(1-\delta)}{\kappa^2(6-2\delta)}$, we have

$$4(1-\sigma)(1-\delta)\sigma_{\min}^2(X^*) \geq 4(6-3\delta)C\|X^*\|_2^2 \geq (6-2\delta)(C^2+3C)\|X^*\|_2^2.$$

Therefore,

$$\langle V,\ \nabla^2 f(X)[V]\rangle \geq 4\sigma(1-\delta)\sigma_{\min}^2(X^*)\|V\|_F^2,$$

which implies that f is L_c-strong convex with $L_c = 4\sigma(1-\delta)\sigma_{\min}^2(X^*)$. □

A.6 Proof of Theorem 4

Proof of Theorem 4. We proceed by mathematical induction to prove this claim. As $X^0 = X^*$ implies $\|X^0H^0 - X^*\|_F = 0 \leq C\|X^*\|_2$. Now, assuming that

$$\|X^kH^k - X^*\|_F \leq C\|X^*\|_2$$

holds for some k, we show that it also holds for $k+1$, namely,

$$\|X^{k+1}H^{k+1} - X^*\|_F \leq C\|X^*\|_2.$$

Note that

$$\begin{array}{rl} & \|X^{k+1}H^{k+1} - X^*\|_F \leq \|X^{k+1}H^k - X^*\|_F = \|(X^k - \eta\nabla f(X^k))H^k - X^*\|_F \\ = & \|X^kH^k - \eta\nabla f(X^kH^k) - X^* + \eta\nabla f(X^*) - \eta\nabla f(X^*)\|_F \\ \leq & \|X^kH^k - \eta\nabla f(X^kH^k) - X^* + \eta\nabla f(X^*)\|_F + \eta\|\nabla f(X^*)\|_F. \end{array} \tag{39}$$

We consider the term $\|\nabla f(X^*)\|_F$ in (39):

$$\begin{aligned}\|\nabla f(X^*)\|_F \quad &= 2\|\mathcal{A}^*[\mathcal{A}(X^*X^{*T}) - \mathcal{A}(X^*X^{*T}) - w]X^* + 2\lambda X^*\|_F \\ &= \|[2\lambda I - 2\mathcal{A}^*(w)]X\|_F \\ &\leq \|(2\lambda I - 2\mathcal{A}^*(w))\|_2\|\|X^*\|_F \\ &\leq 2\sqrt{r^*}(\lambda + \|\mathcal{A}^*(w)\|_2)\|\|\|X^*\|_2.\end{aligned}$$

Next, we focus on the term $\|X^kH^k - \eta\nabla f(X^kH^k) - X^* + \eta\nabla f(X^*)\|_F$ in (39). By the mean value theorem, it follows that

$$\begin{aligned}& X^kH^k - \eta\nabla f(X^kH^k) - X^* + \eta\nabla f(X^*) \\ = \quad & X^kH^k - X^* - \eta(\nabla f(X^kH^k) - \nabla f(X^*)) \\ = \quad & X^kH^k - X^* - \eta\nabla^2 f(X^k(\epsilon))[X^kH^k - X^*],\end{aligned}$$

where $X^k(\epsilon) = X^* + \epsilon(X^kH^k - X^*)$, $\epsilon \in (0, 1)$. Denote $J^k := \nabla^2 f(X^k(\epsilon))$. Then

$$\begin{aligned}& \|X^kH^k - \eta\nabla f(X^kH^k) - X^* + \eta\nabla f(X^*)\|_F^2 \\ = \quad & \langle X^kH^k - X^*, (I - \eta J^k)^2[X^kH^k - X^*]\rangle \\ = \quad & \langle X^kH^k - X^*, (I - 2\eta J^k + (J^k)^2)[X^kH^k - X^*]\rangle.\end{aligned}$$

Since $X^k(\epsilon) \in \mathcal{B}(X^*, C\|X^*\|_2)$, we can choose $\xi \in (\frac{4\sqrt{r^*}\kappa^4(3-\delta)(\|\mathcal{A}^*(w)\|_2+\lambda)}{(1-\delta)^2\|X^*\|_2^2}, 1]$ and $\sigma = \frac{1}{2}$. From this, we get the following inequality:

$$\begin{aligned}3(1+\delta)C\|X^*\|_2^2 = \quad & 3(1+\delta)\tfrac{\xi(1-\sigma)(1-\delta)}{\kappa^2(6-2\delta)}\|X^*\|_2^2 \\ \geq \quad & 3(1+\delta)\tfrac{4\sqrt{r^*}\kappa^4(3-\delta)(\|\mathcal{A}^*(w)\|_2+\lambda)}{(1-\delta)^2\|X^*\|_2^2}\tfrac{(1-\sigma)(1-\delta)}{\kappa^2(6-2\delta)}\|X^*\|_2^2 \\ = \quad & \tfrac{3(1+\delta)\sqrt{r^*}\kappa^2}{(1-\delta)}(\|\mathcal{A}^*(w)\|_2 + \lambda) \\ \geq \quad & \|\mathcal{A}^*(w)\|_2 + \lambda.\end{aligned}$$

Also,

$$\begin{aligned}(3-\delta)C\|X^*\|_2^2 = \quad & (3-\delta)\tfrac{\xi(1-\sigma)(1-\delta)}{\kappa^2(6-2\delta)}\|X^*\|_2^2 \\ \geq \quad & (3-\delta)\tfrac{4\sqrt{r^*}\kappa^4(3-\delta)(\|\mathcal{A}^*(w)\|_2+\lambda)}{(1-\delta)^2\|X^*\|_2^2}\tfrac{(1-\sigma)(1-\delta)}{\kappa^2(6-2\delta)}\|X^*\|_2^2 \\ = \quad & \tfrac{(3-\delta)\sqrt{r^*}\kappa^2}{(1-\delta)}(\|\mathcal{A}^*(w)\|_2 + \lambda) \\ \geq \quad & \|\mathcal{A}^*(w)\|_2.\end{aligned}$$

Due to $\lambda \geq 0$, it holds that $-\|\mathcal{A}^*(w)\|_2 + \lambda \geq -(3-\delta)C\|X^*\|_2^2$. From Lemmas 2 and 3, we conclude

$$\begin{aligned}& \|X^kH^k - \eta\nabla f(X^kH^k) - X^* + \eta\nabla f(X^*)\|_F^2 \\ \leq \quad & \|X^kH^k - X^*\|_F^2 + \eta^2\|J^k\|^2\|X^kH^k - X^*\|_F^2 - 2\eta\langle X^kH^k - X^*, J^k[X^kH^k - X^*]\rangle \\ \leq \quad & \|X^kH^k - X^*\|_F^2 + \eta^2 L_s^2\|X^kH^k - X^*\|_F^2 - 2\eta L_c\|X^kH^k - X^*\|_F^2.\end{aligned}$$

Therefore,

$$\begin{aligned}& \|X^{k+1}H^{k+1} - X^*\|_F \\ \leq \quad & \sqrt{1 + \eta^2 L_s^2 - 2\eta L_c}\|X^kH^k - X^*\|_F + 2\eta\sqrt{r}(\lambda + \|\mathcal{A}^*(w)\|_2)\|X^*\|_2 \\ \leq \quad & \sqrt{1 + \eta^2 L_s^2 - 2\eta L_c}C\|X^*\|_2 + 2\eta\sqrt{r}(\lambda + \|\mathcal{A}^*(w)\|_2)\|X^*\|_2 \\ = \quad & \sqrt{1 + \eta^2 L_s^2 - 2\eta L_c}C\|X^*\|_2 + \tfrac{1}{C}2\eta\sqrt{r}(\lambda + \|\mathcal{A}^*(w)\|_2)C\|X^*\|_2.\end{aligned}$$

Under the step size condition (21), we have

$$\sqrt{1 + \eta^2 L_s^2 - 2\eta L_c} + \frac{1}{C}2\eta\sqrt{r^*}(\lambda + \|\mathcal{A}^*(w)\|_2) \leq 1.$$

Therefore,

$$\|X^{k+1}H^{k+1} - X^*\| \leq C\|X^*\|_2.$$

Finally, it remains to verify that the definition of η is well-defined. It follows from (20) and the choice of ξ and σ that

$$\frac{\xi\sigma(1-\sigma)(1-\delta)^2\|X^*\|_2^2}{\kappa^4(3-\delta)} > \sqrt{r^*}(\|\mathcal{A}^*(w)\|_2 + \lambda).$$

Rearranging the terms, we obtain

$$\frac{\xi(1-\sigma)(1-\delta)}{\kappa^2(6-2\delta)} > \frac{2\sqrt{r^*}(\|\mathcal{A}^*(w)\|_2+\lambda)}{4\sigma(1-\delta)\sigma_{\min}^2(X^*)}.$$

This further implies

$$\frac{\xi(1-\sigma)(1-\delta)}{\kappa^2(6-2\delta)} > \frac{2\sqrt{r^*}(\|\mathcal{A}^*(w)\|_2+\lambda)}{L_c},$$

which, by definition of the constant C, can be rewritten as

$$L_c > \frac{2\sqrt{r^*}(\|\mathcal{A}^*(w)\|_2+\lambda)}{C}.$$

This confirms the well-definedness of η under the stated assumptions. $\square$

A.7 Proof of Lemma 4

Proof of Lemma 4. From (39), we know that $X^kH^k, X^{k+1}H^{k+1} \in \mathcal{B}(X^*, C\|X^*\|_2)$. By Lemma 2 and (Beck, 2017, Lemma 5.7),

$$f(X^{k+1}H^k) \le f(X^kH^k) + \langle \nabla f(X^kH^k), X^{k+1}H^k - X^kH^k\rangle + \frac{L_s}{2}\|X^kH^k - X^{k+1}H^k\|_F^2.$$

Using the facts that $f(XH) = f(X)$ and $\nabla f(XH) = \nabla f(X)H$ for any orthogonal matrix H, we have

$$\begin{aligned} f(X^{k+1}) = \quad & f(X^{k+1}H^k) \\ \le \quad & f(X^k) + \langle \nabla f(X^k), X^{k+1} - X^k\rangle + \tfrac{L_s}{2}\|X^k - X^{k+1}\|_F^2 \\ = \quad & f(X^k) - \langle \nabla f(X^k), \eta\nabla f(X^k)\rangle + \tfrac{L_s}{2}\|\eta\nabla f(X^k)\|_F^2. \end{aligned}$$

When $\eta \le \frac{1}{L_s}$, it holds that

$$f(X^{k+1}) \le f(X^k) - \frac{\eta}{2}\|\nabla f(X^k)\|_F^2.$$

$\square$

A.8 Proof of Lemma 5

Proof of Lemma 5. First, note that f is bounded below. By Lemma 4, the sequence $f(X^kH^k)$ satisfies

$$f(X^{k+1}H^{k+1}) \le f(X^kH^k) - \frac{\eta}{2}\|\nabla f(X^kH^k)(H^k)^T\|_F^2,$$

which implies that $f(X^kH^k)$ is monotonically decreasing and thus convergent. Therefore,

$$\lim_{k\to\infty}\|\nabla f(X^kH^k)\|_F^2 = \lim_{k\to\infty}\|\nabla f(X^kH^k)(H^k)^T\|_F^2 = 0.$$

Moreover, since $\{X^kH^k\} \subseteq \mathcal{B}(X^*, C\|X^*\|_2)$, and $\mathcal{B}(X^*, C\|X^*\|_2)$ is closed and bounded (hence compact), the sequence $\{X^kH^k\}$ has at least one accumulation point $\bar{X}$ in $\mathcal{B}(X^*, C\|X^*\|_2)$. By the continuity of ∇f, we have $\nabla f(\bar{X}) = 0$, and thus $\bar{X}$ is a stationary point of the non-convex problem (5). $\square$

A.9 Proof of Theorem 5

Proof of Theorem 5. From Lemma 5, we know that $\bar{X} \in \mathcal{B}(X^*, C\|X^*\|_2)$. Then

$$\begin{aligned} \|\bar{M} - M^*\|_F = \quad & \|\bar{X}\bar{X}^T - X^*X^{*T}\| \le \|\bar{X} - X^*\|_F^2 + 2\|X^*\|_2\|\bar{X} - X^*\|_F \\ \le \quad & C^2\|X^*\|_2^2 + 2C\|X^*\|_2^2. \end{aligned} \tag{40}$$

Substituting

$$C = \frac{\xi(1-\sigma)(1-\delta)}{\kappa^2(6-2\delta)} \text{ with } \xi \to \frac{4\sqrt{r}\kappa^4(3-\delta)(\|\mathcal{A}^*(w)\|_2+\lambda)}{(1-\delta)^2\|X^*\|_2^2}, \text{and } \sigma = \frac{1}{2}$$

into (40) yields the claimed result. $\square$