Structured Noise to Help Non-Convexity: Solving Matrix Completion as Noisy Matrix Sensing

Ziye Ma [1]

Abstract

The training of all modern machine learning models including deep neural networks and large languages models can be considered as solving non-convex optimization problems, which are in general NP-Hard and notoriously difficult to solve efficiently. In this work, we propose a new approach to tackle non-convexity by introducing structured noise in order to smooth out the challenging optimization landscape at the cost of some accuracy. We demonstrate our approach using matrix completion, a key non-convex problem critical to various machine learning applications like recommender systems. Conventionally, a benign optimization landscape, and by extension, the successful recovery of ground truth for matrix completion, can only be guaranteed under stringent conditions, such as the presence of strong incoherence and large number of observations. By introducing a specific kind of perturbation, we could transform matrix completion problems into noisy matrix sensing problems, which in turn allows the use of over-parametrization to achieve guarantees of recovery without the need for restrictive assumptions. This novel strategy not only pioneers the solving of matrix completion, but also opens new pathways for addressing non-convex challenges globally, potentially benefitting machine learning practices more broadly.

1. Introduction

Non-convex optimization presents significant challenges in modern machine learning, particularly in training complex models like deep neural networks, generative models, and beyond. Unlike convex problems, non-convex optimization landscapes are characterized by multiple local minima, saddle points, and regions of flat curvature, complicating the search for global optima (Jain and Kar, 2017). This complexity often leads to convergence issues, where algorithms may become trapped in suboptimal solutions, hindering model performance and generalization capabilities. Traditional gradient-based methods, such as stochastic gradient descent (SGD) and its adaptive variants like Adam and RMSProp, are commonly employed to navigate these complex landscapes. These methods iteratively adjust model parameters to minimize loss functions, but their efficiency can be compromised by the intricate topography of non-convex spaces. To mitigate these challenges, techniques like learning rate scheduling, momentum, and gradient clipping are often utilized to enhance convergence and stability during training (Fotopoulos et al., 2024).

However, while these techniques offer partial improvements, they do not fully address the underlying difficulties posed by non-convex optimization (Sun and Luo, 2019). The reliance on local heuristics and adjustments does not fundamentally alter the non-convex nature of the problem. Current methods are far from solving non-convex training effectively, as they often do not guarantee convergence to global optima and can lead to varied results depending on the initial conditions and specific configurations used. Consequently, a more in-depth and comprehensive study of such landscapes is essential. One popular way of doing so is to focus on mathematically structured non-convex problems like low-rank matrix recovery. Such problems have garnered increased attention due to their potential to provide deep insights. Low-rank matrix tasks, including matrix completion (MC) and matrix sensing (MS), are crucial in numerous domains like machine learning and signal processing. They involve reconstructing a low-rank matrix from incomplete observations or linear measurements, with applications spanning collaborative filtering in recommendation systems (Koren et al., 2009), motion detection (Fattahi and Sojoudi, 2020), and power system state estimation (Zhang et al., 2017; Jin et al., 2019), to image recovery (Gu et al., 2014) and biomedical imaging (Lustig et al., 2008). More significantly, as these frameworks can encapsulate any polynomial optimization problem (Molybog et al., 2020) and are equivalent to training two-layer quadratic neural networks (Li et al., 2018), their theoretical impact extends well beyond their direct applications in the broader machine learning community.

[1]Department of Computer Science, City University of Hong Kong. Correspondence to: Ziye Ma <ziyema@cityu.edu.hk>.

Preprint.

Matrix sensing generally involves recovering a matrix from a set of linear measurements, formulated as:

$$\min_{X \in \mathbb{R}^{n \times r_{\text{search}}}} \frac{1}{2}\|\mathcal{A}(XX^T) - b\|_2^2 := f(XX^T) \quad \text{(MS)} \quad (1)$$

For the sake of convenience, we also denote $h(X) := f(XX^\top)$. Here, $\mathcal{A}$ acts on the low-rank matrix XX^T (rank bounded by r_{search} by construct) and compares it to a vector of observations $b = \mathcal{A}(M^*)$, with M^* being the rank-r ground truth matrix of interest. $\mathcal{A}(\cdot) : \mathbb{R}^{n\times n} \mapsto \mathbb{R}^m$ is a linear function defined as $\mathcal{A}(M) = [\langle A_1, M\rangle, \ldots, \langle A_m, M\rangle]^T$ where $\langle A_i, M\rangle := \operatorname{tr}(A_i^\top M)$ and the sensing matrices $\{A_i\}_{i=1}^m$ are given. For simplicity, we assume $r_{\text{search}} = r$. The matrix completion challenge, a special case of matrix sensing, is given by:

$$\min_{X \in \mathbb{R}^{n \times r}} \frac{1}{2}\|\mathcal{A}_\Omega(XX^T - M^*)\|_2^2 \quad \text{(MC)} \qquad (2)$$

where $\Omega \subseteq [n] \times [n]$ represents the observed entries of an $n \times n$ matrix. We use the notation N_Ω to denote the matrix where

$$(N_\Omega)_{i,j} = N_{i,j} \cdot \mathbf{1}_{(i,j)\in\Omega} \qquad (3)$$

for any arbitrary $N \in \mathbb{R}^{n\times n}$. $\mathcal{A}_\Omega(\cdot)$ is used to specifically denote the sensing operator of the matrix completion problem, where $\mathcal{A}_\Omega(M) = \operatorname{vec}(M_\Omega)$.

Matrix completion is distinguished by its reliance on the sample rate and the matrix's incoherence parameters. These parameters dictate the spread of matrix information across its entries and singular vectors (Candès and Recht, 2009; Candès and Tao, 2010). This dependency complicates matrix completion compared to matrix sensing, where challenges are often more tractable due to properties like Restricted Isometry Property (RIP) (Recht et al., 2010). In this work, we use the equivalent notions of Restricted Strong Convexity (RSC) and Smoothness (RSS) to offer greater flexibility:

Definition 1.1 (Restricted Strong Smoothness (RSS) and Restricted Strong Convexity (RSC))**.** The linear operator $\mathcal{A} : \mathbb{R}^{n\times n} \mapsto \mathbb{R}^m$ satisfies the (L_s, r)-RSS property and the (α_s, r)-RSC property if

$$\mathcal{A}(M) - \mathcal{A}(N) \le \langle M - N, \nabla f(N)\rangle + \frac{L_s}{2}\|M - N\|_F^2$$
$$\mathcal{A}(M) - \mathcal{A}(N) \ge \langle M - N, \nabla f(N)\rangle + \frac{\alpha_s}{2}\|M - N\|_F^2$$

are satisfied, respectively for all $M, N \in \mathbb{R}^n$ with $\operatorname{rank}(M), \operatorname{rank}(N) \le r$. Note that RSS and RSC provide a more expressible way to represent the RIP property (Recht et al., 2010), with $\delta_r = (L_s - \alpha_s)/(L_s + \alpha_s)$.

Matrix completion problems, on the other hand, all have missing entries, which means that they could not have a valid RSC constant since the null-space of $\mathcal{A}_\Omega(\cdot)$ is always non-trivial. This is why incoherence condition was proposed to use as a metric to guarantee recovery of completion problems:

Definition 1.2 (μ_0-incoherence)**.** (Ge et al., 2017) Given a rank-r matrix $M \in \mathbb{R}^{n_1\times n_2}$, we say it is μ_0-incoherent if its truncated SVD decomposition $U\Sigma V^\top$ satisfies

$$\|e_i^\top U\|_2 \le \sqrt{\mu_0 r/n_1}, \;\; \|e_j^\top V\|_2 \le \sqrt{\mu_0 r/n_2}$$

$\forall i, j \in [n_1], [n_2]$, where e_i is the i-th standard basis of $\mathbb{R}^{n_1}$ and e_j is the j-th standard basis of $\mathbb{R}^{n_2}$.

Since μ_0 is hard to gauge prior to solving this problem, applying matrix completion guarantees can be more challenging than matrix sensing problems with valid RSS and RSC constants. Thus, this paper introduces novel methodologies to solve matrix completion, thereby broadening its theoretical accessibility and enhancing its practical applicability. The proposed approaches aim to reduce the reliance on strong assumptions, making these powerful techniques more accessible to a wider range of real-world applications.

1.1. Related Works

We briefly review some notable prior works dedicated to solving matrix sensing and matrix completion problems with guarantees, and highlight the (surprising) power of over-parametrization that is attracting increased attention in machine learning community.

Recovery Guarantees

The foundational work by (Candès and Recht, 2009) established that exact matrix recovery is possible from few entries, requiring a sample size of $\mu_0 n^{1.2} r \log(n)$ for $n \times n$ matrices of rank r with incoherence parameter μ_0. Enhancements in recovery guarantees and computational efficiencies followed, including spectral-gradient descent algorithms by (Keshavan et al., 2010) and deeper insights into incoherence by (Candès and Tao, 2010). Studies by (Recht, 2011) and (Gross, 2011) expanded on these by demonstrating successful recovery without uniform random sampling, while (Ding and Chen, 2020) refined sampling orders further to $\mu_0 r \log(\mu_0 r) n \log(n)$.

Research into Burer-Monteiro factorization has explored non-convex optimization strategies for matrix completion, including greedy algorithms (Lee and Bresler, 2010; Wang et al., 2014), alternating minimization (Haldar and Hernando, 2009; Tanner and Wei, 2016; Wen et al., 2012), iterative thresholding (Klopp, 2015) and Riemannian optimization (Mishra et al., 2014; Dai and Milenkovic, 2010)—reviewed comprehensively in (Nguyen et al., 2019). Although lacking explicit recovery guarantees, these methods demonstrate empirical effectiveness, with some, like ADMiRA

(Lee and Bresler, 2010), dependent on RIP conditions not generally applicable in matrix completion. Another interesting line of work converts inductive matrix completion into regular MC (Ghassemi et al., 2018; Zilber and Nadler, 2022), but they have extra information A, B not considered in this classical setting (2) and they also assumed incoherence conditions for A, B matrices, which we hope to avoid in this work.

Recent developments (Ge et al., 2016; 2017; Du et al., 2017) have provided robust recovery guarantees for matrix completion using gradient descent and variants. These studies confirm that absent spurious solutions if each entry is observed with a probability $p \geq \text{poly}(\kappa, r, \mu_0, \log n)/n$, ensuring the success of BM in polynomial time with saddle-escaping algorithms, reflecting SDP literature findings. Similarly to MC results, it has long been known (Recht et al., 2010; Candès and Tao, 2010) that the RIP constants (see Definition 1.1) play a central role in determining whether this non-convex problem could be solved to optimality with guarantees. It is widely understood that $\delta_{2r} = 1/2$ is a sharp threshold for the factorized Burer-Monteiro (BM) formulation (1) (Zhang et al., 2021; Ma et al., 2022), and a sufficient bound for SDP relaxation (Cai and Zhang, 2013).

This raises the question of applying matrix sensing's RIP-based literature to matrix completion, an area that remains largely unexplored despite initial efforts like (Zhang et al., 2023) to bridge this theoretical gap.

Power of Over-parametrization

Recent studies have highlighted over-parametrization as a crucial strategy in matrix sensing when RIP constants are suboptimal (i.e., $\delta_{2r} \geq 1/2$). Research by (Zhang, 2021; 2022) examined cases where the search rank r_{search} exceeds the true rank r, thus increasing the problem's parametrization. (Zhang, 2022) demonstrated that for $r_{\text{search}} > r[(1+\delta_n)/(1-\delta_n) - 1]^2/4$ and $r \leq r_{\text{search}} < n$, each solution $\hat{X}$ satisfies $\hat{X}\hat{X}^\top = M^*$. Similarly, (Ma and Fattahi, 2022) established analogous results under RIP-type conditions for the ℓ_1 loss. In addition to the classic over-parametrization strategy of increasing r_{search}, there are many other forms of over-parametrization, and convex relaxation (Recht et al., 2010) is certainly one of them since the parameter space is increased from $\mathcal{O}(nr)$ to $\mathcal{O}(n^2)$. Regarding convex relaxation, Yalcin et al. (2023) showed that the RIP threshold for exact recovery using SDP can approach 1 when M^* has a high true rank, thus underscoring the efficacy of over-parametrization. Nevertheless, the practical applicability of these conditions is limited, leading (Ma et al., 2023) to explore an alternative approach to over-parametrization by lifting the search space of (1) into general tensor space and bank on important concepts from Sums-of-Squares optimization (Parrilo, 2003; Lasserre, 2001) to convert spurious local minimizers of (1) into strict saddle points in the lifted space so that they could be escaped by modern optimizers. Despite its utility in resolving spurious solutions, this tensor approach's applicability to matrix completion remains constrained by the need for a valid RIP constant.

1.2. Our Approach and Main Contributions

In an effort to bridge the theoretical gaps identified in matrix completion (MC) problems, our research introduces a framework designed to carefully perturb the problem so that they will exhibit Restricted Isometry Property (RIP) characteristics, albeit with a trade-off in solution precision. The core of our methodology involves making the nullspace of our sensing matrices to be trivial and leveraging the power of over-parametrization in problem solving. Here are the principal steps of our approach:

1. **Constructing Surrogate Problem:** We construct a surrogate problem (ϵ-MC) to solve by slightly changing $\mathcal{A}_\Omega$. This is achieved by introducing controlled perturbations to the sensing matrices, transforming a MC scenario into a manageable noisy MS problem. This step is crucial for aligning the MC problem with the more favorable theoretical properties of MS.

2. **Verifying Surrogate Quality:** We establish that the global solution to the surrogate problem will be close to the ground truth M^* with high probability under mild assumptions.

3. **Adaptation of Lifted Tensor Framework:** We extend the lifted tensor framework, originally discussed in (Ma et al., 2024), to our perturbed MS problem. This is because we need the power of over-parametrization to handle the perturbed MS problem with very high RIP constants.

While the details of these steps might appear counterintuitive at first glance, we will provide a thorough exposition in subsequent sections to clarify our methods and findings. Moreover, this strategy leads to two significant contributions to the field of low-rank matrix recovery:

- Proposes a framework to perform matrix completion with global guarantees without the need for the ground truth to obey incoherence conditions or for the observed entries to have certain structures, enabling a much wider range of MC problems to be solved with guarantees.

- We validate that the lifted tensor framework (Ma et al., 2023; 2024) remains effective in scenarios with noise corruption, thereby expanding its utility and robustness.

2. Notation

Scalar values such as $\sigma_i(M)$ and $\lambda_i(M)$ represent the i-th largest singular value and eigenvalue of matrix M, respectively. The inner product between two matrices $\langle A, B\rangle$ is defined as $\mathrm{tr}(A^\top B)$. The Euclidean norm of a vector v is denoted as $\|v\|$, while $\|M\|_F$ and $\|M\|_2$ are used for the Frobenius and induced l_2 norms of a matrix M. Vectorization $\mathrm{vec}(M)$ stacks the columns of M into a vector, where $\mathrm{mat}(v)$ is its reserve operation. The set of integers from 1 to n is expressed as $[n]$. Moreover, $\circ l$ indicates a repeated Cartesian product for l times, $\oslash$ refers to the Kronecker product, and $\otimes$ signifies the tensor outer product. If these notations come with subscripts, they denote the dimension along which the operation is performed. Finally, if $S \in [n] \times [m]$ represents a subset of indices of a $n \times m$ matrix, then N_S refers to the sub-matrix of $N \in \mathbb{R}^{n\times m}$ relevant to S as per (3), and $\|N\|_{S,F}$ denotes the Frobenius norm of N_S.

3. The Perturbed Matrix Completion Formulation

As explained above, most literature regarding the recovery guarantees of matrix sensing problems require some valid RIP (RSC and RSS) constant. However, the attainment of such a constant automatically implies a trivial nullspace, meaning that $\mathcal{A}$ only maps a zero matrix to a zero vector, which is impossible for matrix completion problems. To demonstrate why this is, let's consider a 2×2 matrix recovery problem, and say we observed three entries of some $M \in \mathbb{R}^{2\times 2}$ except for the lower-right entry. This will correspond to the case where

$$\begin{aligned}\mathcal{A}_\Omega(M) = \mathrm{vec}(\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} \odot M) = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}\mathrm{vec}(M)\\ := T_\Omega \,\mathrm{vec}(M) \in \mathbb{R}^4\end{aligned} \tag{4}$$

For this $\mathcal{A}_\Omega$ to exhibit any RIP constant $\delta < 1$, it is required that $\|\mathcal{A}_\Omega(M)\|_2^2 \geq (1-\delta)\|M\|_F^2$, meaning that the T_Ω matrix above cannot output 0 unless M is a zero matrix. Nevertheless, for this specific example, even if we observed three out of four entries, we can simply set $\mathrm{vec}(M) = [0,0,0,1]^\top$ to make $\mathcal{A}_\Omega(M) = 0$, violating the RIP condition. This is a simple example showing us that RIP condition will not hold for matrix completion problems unless all entries are observed, regardless of its size. Therefore, it begs the question of *whether we could use the better studied, more powerful over-parametrized MS framework to offer guarantees for MC problems?*

Despite this limitation, a surprisingly simple solution exists. The primary issue is the zero entries in the diagonal of T_Ω, contributing to a non-trivial nullspace. By perturbing these zero entries slightly with a small number $\epsilon \in (0,1]$, we can eliminate the nullspace. Revisiting (4), consider a perturbed sensing operator $\mathcal{A}_{\Omega,\epsilon}$:

$$\begin{aligned}\mathcal{A}_{\Omega,\epsilon}(M) = \mathrm{vec}\left(\begin{bmatrix}1 & 1\\ 1 & \epsilon\end{bmatrix} \odot M\right) = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&\epsilon\end{bmatrix}\mathrm{vec}(M)\\ := T_{\Omega,\epsilon}\,\mathrm{vec}(M)\end{aligned} \tag{5}$$

in which $T_{\Omega,\epsilon}$ has a trivial nullspace as promised. However, different operators lead to different observations. For example, when considering the case of (4), using $\mathcal{A}_\Omega$ and $\mathcal{A}_{\Omega,\epsilon}$ on the same matrix results in:

$$\mathcal{A}_\Omega(M) = \begin{bmatrix}M_{1,1}\\M_{1,2}\\M_{2,1}\\0\end{bmatrix} \longrightarrow \mathcal{A}_{\Omega,\epsilon}(M) = \begin{bmatrix}M_{1,1}\\M_{1,2}\\M_{2,1}\\\epsilon M_{2,2}\end{bmatrix}$$

Therefore, the original observation b can be considered a noisy observation under the context of $\mathcal{A}_{\Omega,\epsilon}$, with

$$b = \mathcal{A}_{\Omega,\epsilon}(M^*) + w_\epsilon, \;\; w_\epsilon = \begin{bmatrix}0 & 0 & 0 & -\epsilon M^*_{2,2}\end{bmatrix}^\top \tag{6}$$

where $w_\epsilon \in \mathbb{R}^{n^2}$ can be considered a noise term. With this idea in place, we formally introduce our perturbed MC problem to solve:

$$\min_{X\in\mathbb{R}^{n\times r_{\text{search}}}} \|\mathcal{A}_{\Omega,\epsilon}(XX^T) - b\|_2^2 := f_{w_\epsilon}(XX^\top) \quad (\epsilon\text{-MC}) \tag{7}$$

Essentially, we're transforming a noiseless matrix completion problem into a noisy matrix sensing problem with operator $\mathcal{A}_{\Omega,\epsilon}$ and deterministic noise w_ϵ. This approach, notably, ensures the attainment of valid RSS/RSC parameters, equivalent to RIP constants.

Lemma 3.1. *Given an arbitrary matrix completion problem with sensing operator $\mathcal{A}_\Omega$, if this operator is perturbed to produce $\mathcal{A}_{\Omega,\epsilon}$ according to (5) with a scalar $\epsilon \in (0,1]$, then the ϵ-MC problem will exhibit $(1,n)$-RSS property and the (ϵ^2,n)-RSC property.*

The proof is straightforward and omitted for brevity. With that said, another major challenge that the perturbed formulation of ϵ-MC problems brings is that the global solution of ϵ-MC might not be M^* anymore. This can be easily seen since $\|\mathcal{A}_{\Omega,\epsilon}(M^*) - b\|_2^2 \neq 0$. In other words, since the core idea of our approach is to solve the surrogate ϵ-MC problem in order recover M^*, we need to know the conditions under which M^* will be close to the global solution of (7), since later on we will show that over-parametrization via lifting could help us reach the global solution, denoted as $M^\dagger$, with guarantees. Inspired by (Ma and Fattahi, 2023), we hope to link it with the number of corrupted observations. If we adopt the standard assumption that each entry of the matrix is independently observed with probability p, then we could generalize this observation by linking it to p. As our next step, we show that $M^\dagger$ will be very close to M^* with high probability, and we can further achieve a tradeoff between sample rate p and geometric uniformity captured by ϵ.

We will briefly go over the high-level ideas in this derivation and present our formal theorem in the end. Since we assumed that $M^\dagger$ is the global optimum of (7), then by definition it gives that $f_{w_\epsilon}(M^\dagger)-f_{w_\epsilon}(M^*) \leq 0$. If we partition the set Ω into $\bar{S}$, the observed, noiseless entries, and S, the unobserved, perturbed entries, then we could decompose $f_{w_\epsilon}(M^\dagger) - f_{w_\epsilon}(M^*)$ as:

$$\begin{aligned} f_{w_\epsilon}(M^\dagger) - f_{w_\epsilon}(M^*) &= \frac{1}{2}\|\mathcal{A}_{\Omega,\epsilon}(M^\dagger - M^*)\|_{\bar{S},2}^2 \\ &+ \frac{1}{2}\|\mathcal{A}_{\Omega,\epsilon}(M^\dagger - M^*) - w_\epsilon\|_{S,2}^2 - \frac{1}{2}\|w_\epsilon\|_{S,2}^2 \\ \geq &\frac{1}{2}\|\mathcal{A}_{\Omega,\epsilon}(M^\dagger - M^*)\|_{\bar{S},2}^2 - \frac{1}{2}\epsilon^2\|M^*\|_{S,F}^2 \end{aligned} \tag{8}$$

where $\|\cdot\|_{S,2}$ denotes the $l2$ norm of the sub-vector with entries in set S. Then if we add $\frac{1}{2}\|M^\dagger - M^*\|_{S,2}^2$ to both sides of (8), it is easy to show

$$\frac{1}{2}\|M^\dagger - M^*\|_F^2 \leq \frac{1}{2}\left(\|M^\dagger - M^*\|_{S,F}^2 + \epsilon^2\|M^*\|_{S,F}^2\right) \tag{9}$$

Here we look into the right hand side terms of (9) a bit more carefully, and realize that both $\|M^\dagger - M^*\|_{S,F}^2$ and $\|M^*\|_{S,F}^2$ are random variables with their sizes dependent on the sampling rate. Since $\|\cdot\|_S^2$ only denotes the size of the sub-matrix that are not observed (therefore perturbed by ϵ), if our sample rate p is small, this norm would also be small in expectation. Combining this intuition with concentration inequality to control deviation, gives this next theorem which serves as our main result showing why the ϵ-MC problem (7) can serve as a meaningful surrogate to the original MC problem.

Theorem 3.2. *Assume that $M^\dagger \in \mathbb{R}^{n\times n}$ is a symmetric, rank-r matrix that is a global optimum of (7) with an $\epsilon \in (0,1]$. Assume that each entry of the original MC problem is independently observed with probability p, then for any $\chi \leq p \in \mathbb{R}$,*

$$\|M^\dagger - M^*\|_F^2 \leq \frac{1-p+\chi}{p-\chi}\epsilon^2\|M^*\|_F^2 \tag{10}$$

holds with probability at least $1 - \exp\left(-2\chi^2\|d\|_1^2/\|d\|_2^2\right)$, where $d \in \mathbb{R}^{n^2}$ is defined as

$$\operatorname{vec}(M^\dagger - M^*) \odot \operatorname{vec}(M^\dagger - M^*) + \epsilon^2 \operatorname{vec}(M^*) \odot \operatorname{vec}(M^*) \tag{11}$$

We begin by noting that for any vector d, elementary inequalities ensure that $1 \leq \|d\|_1^2/\|d\|_2^2 \leq n^2$. This ratio increases as the values of d become more evenly distributed. In proving our theorem, we employed Hoeffding's inequality to achieve clear and interpretable results. While other concentration inequalities like Bennett's inequality can also be applied to independent, bounded variables, they do not always provide a tighter bound and would complicate the expression, hence they are not included in this work. We recognize the potential for employing more advanced statistical tools to refine these bounds. Readers interested in exploring this further can find the proof of the theorem in the Appendix.

4. Lifted Tensor Framework with Noise

Now that we are able to reformulate the original MC problem (2) into the new ϵ-MC problem (7), it presents us with a new challenge. Although now (7) admits valid RSC/RSS constants, this is nevertheless still a difficult matrix sensing problem to solve due to its small RSC constant (or large RIP constant). Thus, it is important that we apply an over-parametrized framework to deal with it in order to compensate for the poor geometric uniformity.

To this end, we employ the lifted tensor framework proposed in (Ma et al., 2023), since it has the ability to deal with really small RSC constants, like those we have in ϵ-MC. However, in their original work, measurements were assumed to be clean, and this is incompatible with our framework since we hope to deal with noisy MS problems. Thus, we generalize the original results in (Ma et al., 2023), and also in its subsequent work (Ma et al., 2024) to demonstrate how the inclusion of noise could affect guarantees in when using a higher-order tensor parametrization.

First of all, we present the lifted tensor problem when our observations are corrupted by some random noise $\tilde{w} \in \mathbb{R}^m$,

$$\min_{\mathbf{w}\in\mathbb{R}^{nrol}} \quad \|\langle \mathbf{A}^{\otimes l}, \langle \mathbf{P}(\mathbf{w}), \mathbf{P}(\mathbf{w})\rangle_{2*[l]}\rangle - \tilde{b}^{\otimes l}\|_F^2 \tag{12}$$

where $\tilde{b} = \mathcal{A}(M^*) + \tilde{w}$, and $\mathbf{A} \in \mathbb{R}^{m\times n\times n}$ is a three-way tensor which can be seen as a concatenation of all sensing matrices $\{A_i\}_{i=1}^m$, and $\mathbf{w} \in \mathbb{R}^{nrol}$ is the tensor decision variable used to increase the parametrization of $X \in \mathbb{R}^{n\times r}$. Here, ${}^{\otimes l}$ simply denotes l times of repeated tensor outer product, and $\mathbf{P}$ is just another constant permutation tensor used for correct multiplication. The gist of this paper is not on the tensor formulation, thus many details are deferred to Appendix A, and interested readers can learn more about general tensor knowledge and problem details there. For convenience's sake, we define $f^l(\cdot) : \mathbb{R}^{no2l} \mapsto \mathbb{R}$ and $h^l(\cdot) : \mathbb{R}^{[n\times r]ol} \mapsto \mathbb{R}$ as $f^l(\mathbf{M}) := \|\langle \mathbf{A}^{\otimes l}, \mathbf{M}\rangle - \tilde{b}^{\otimes l}\|_F^2$ and $h^l(\mathbf{w}) = f^l(\langle \mathbf{w}, \mathbf{w}\rangle_{2*[l]})$, with $\nabla f^l(\cdot) = \nabla_{\mathbf{M}} f^l(\cdot)$ and $\nabla h^l(\cdot) = \nabla_{\mathbf{w}} h^l(\cdot)$.

In the original works, it was proven that the lifted formulation (12) is able to convert spurious solutions in (1) to strict saddle points via its drastic over-parametrization if this spurious solution is somehow far away from the ground truth M^*. However, the first thing to note here is that for any corrupted MS problem (its observation b is not clean and affected by noise), its global solution might not correspond to M^* anymore, which is the same challenge that we faced in the ϵ-MC formulation. This means that spurious solutions have to be even more distant to M^* for it be converted into strict saddles, depending on the intensity of noise. The result is summarized in the following theorem:

Theorem 4.1. *Consider an arbitrary second-order point $\hat{X} \in \mathbb{R}^{n\times r}$ of the factorized matrix sensing objective in the form of (1) where its observations b could be potentially corrupted by some random noise $\tilde{w} \in \mathbb{R}^m$ (i.e. $b = \tilde{b}$). Assuming that the linear operator $\mathcal{A}(\cdot)$ in (1) satisfies the RSC and RSS conditions with constants α_s, L_s respectively. Then $\hat{\mathbf{w}} = \operatorname{vec}(\hat{X})^{\otimes l}$ is a strict saddle of (12) with a rank-1 symmetric escape direction if*

$$\|M^* - \hat{X}\hat{X}^\top\|_F^2 \geq \frac{L_s}{\alpha_s}\lambda_r(\hat{X}\hat{X}^\top)\operatorname{tr}(M^*) + \frac{\|\tilde{w}\|_2^2}{\alpha_s} \quad (13)$$

with an odd l satisfying

$$l > \frac{1}{1-\log_2(2\beta)}, \quad \beta := \frac{L_s \operatorname{tr}(M^*)\lambda_r(\hat{X}\hat{X}^\top)}{\alpha_s\|M^* - \hat{X}\hat{X}^\top\|_F^2 - \|\tilde{w}\|_2^2}. \quad (14)$$

The proof of this theorem is located in Appendix B. The theorem highlights how the conversion radius from spurious solutions to strict saddles is influenced by the norm of the noise $\tilde{w}$. Setting $\tilde{w} = 0$ allows this theorem to coincide with Theorem 4 from (Ma et al., 2024). More importantly, it is crucial for the critical point $\hat{\mathbf{w}}$ in (12) to be a rank-1 tensor to possess a negative escape direction. For a detailed definition of tensor rank, please see Appendix A. According to (Ma et al., 2024), employing a gradient descent (GD) algorithm with sufficiently small initialization ensures that the search is conducted over approximately rank-1 tensors throughout the GD trajectory. This work further establishes that this characteristic remains unchanged when b is substituted with $\tilde{b}$, showing that the effects of implicit bias induced by vanilla GD is agnostic of noise. Here we present an informal version of this result to facilitate understanding, and the full version and its proof can be found in Appendix A and Appendix B respectively:

Theorem 4.2 (Informal)**.** *Consider a finite-horizon gradient descent trajectory $\{\mathbf{w}_t\}_{t\in[T]}$ of (12) with $\mathbf{w}_{t+1} = \mathbf{w}_t - \eta\nabla h^l(\mathbf{w}_t)$ starting from the initialization $\mathbf{w}_0 = \xi x_0^{\otimes l}$ with $\xi \in \mathbb{R}$ denoting the scale of the initialization, η representing the step-size and $x_0 \in \mathbb{R}^{nr}$ being an arbitrary vector with $\|x_0\|_2^2 = 1$. Then for a sufficiently small ξ, there exists an iteration number $t(\kappa, l) \geq 1$ that depends on an arbitrary constant $\kappa < 1$ and lifting degree l such that*

$$\frac{\lambda_2^v(\mathbf{w}_t)}{\lambda_1^v(\mathbf{w}_t)} \leq \kappa, \qquad \forall t \in [t(\kappa, l), T] \quad (15)$$

where $\lambda_i^v(\cdot)$ denotes the i-th largest v-eigenvalue (see Appendix A) of a given tensor, thereby meaning that the ratio between the second largest eigenvalue and the largest eigenvalue is small, which implies that the tensor is approximately rank-1 after iteration $t(\kappa, l)$ along the GD trajectory $\{\mathbf{w}_t\}_{t\in[T]}$. Furthermore, $t(\kappa, l)$ increases with a smaller κ, meaning that the tensor along the trajectory will become increasingly like rank-1 as GD updates happen.

If x_0 is initialized as $x_0 = v_1 + g \in \mathbb{R}^{nr}$ where g is i.i.d. centered Gaussian and v_1 is the first singular vector of U, where U is a function of $\mathcal{A}$ and b; we can also show that $\mathbf{w}_t$ will be approximately rank-1 as soon as $t \asymp \ln(1/\kappa)\ln\left((1+\eta\sigma_1^l(U))/(1+\eta\sigma_2^l(U))\right)^{-1}$, if ξ is chosen as a function of U, r, n, L_s, without the need for it to be arbitrarily small. However, since such results are not the main focal point of this work, we will not elaborate here for the sake of succinctness. The main takeaway is that by incorporating the noise $\tilde{w}$ into our objective (12), the ability of gradient descent algorithms to induce implicit bias remains unchanged. It is also worth noting that the results presented in this subsection applies to all tensor problems in the form of (12), which are lifted from general noisy matrix sensing problems, and not specific to our ϵ-MC problem.

5. Main Results

Our goal is to achieve a globally optimal solution for the ϵ-MC problem because it closely represents the M^* solution. However, this becomes challenging due to the α_s constant in equation (7), which depends on the small value of ϵ. To address this, rather than solving the problem using its basic matrix (BM) factorized form (as shown in equation (1)), which lacks global optimization guarantees, we apply more complex techniques with over-parametrization. We previously demonstrated that the lifted tensor framework (12), independent of the specific ϵ-MC problem, effectively handles noise in the observed data (when b becomes $\tilde{b}$), with the quality of the guarantee degrading as the magnitude of corruption increases.

By integrating these methodologies, we demonstrate a new way to approximately solve the generic MC problem (as formulated in equation (2)) while still providing reliable global solutions, as elaborated in our main theorem below

Theorem 5.1. *Consider the matrix completion problem of completing a $n \times n$, rank-r matrix M^*, where $\Omega \subseteq [n] \times [n]$ denotes the set of observed entries and $\bar{\Omega}$ denotes the unobserved entries. Introduce a perturbation $\epsilon \in (0, 1]$ to formulate an ϵ-MC problem as per (7). Applying the*

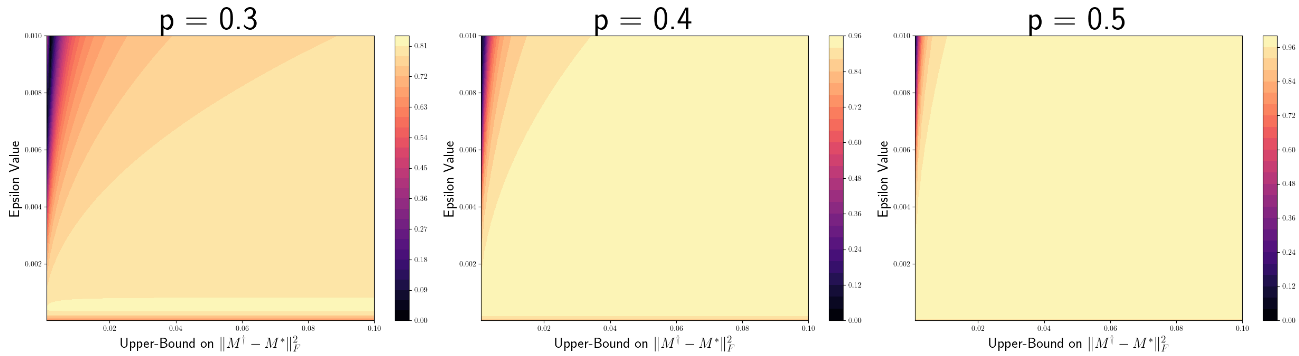


Figure 1. Probability Lower-Bound for Theorem 3.2.

tensor framework described in (12) *to this ϵ-MC problem yields the following results:*

For any rank-1 critical point $\hat{\mathbf{w}} = \text{vec}(\hat{X})^{\otimes l}$ of (12)*, if it is a second-order point (local minima), this implies that*

$$\|M^* - \hat{X}\hat{X}^\top\|_F < \frac{1}{\epsilon}\lambda_r(\hat{X})\sqrt{\text{tr}(M^*)} + e_1 \tag{16}$$

holds with probability at least q, under the condition that l is odd and meets the requirement:

$$l > \frac{1}{1 - \log_2(2\beta)}, \quad \beta := \frac{\text{tr}(M^*)\lambda_r(\hat{X}\hat{X}^\top)}{\epsilon^2\left(\|M^* - \hat{X}\hat{X}^\top\|_F^2 - e_2\right)}. \tag{17}$$

For all instances of the MC problem, the following hold:

$$e_1 = e_2 = \|M^*\|_{\bar{\Omega},F}^2, \quad q = 1 \tag{18}$$

Alternatively, if all entries are observed independently with probability p, e_2 could be ignored (i.e. $e_2 = 0$) and:

$$e_1 = \sqrt{\frac{1 - p + \chi}{p - \chi}}\epsilon\|M^*\|_F, \; q = 1 - e^{\left(-2\chi^2\|d\|_1^2/\|d\|_2^2\right)} \tag{19}$$

where χ and d are defined as per Theorem 3.2.

Our main theorem builds directly on the results of Theorem 4.1, applying specific parameters ($L_s = 1$, $\alpha_s = \epsilon^2$) along with the definition of w_ϵ from equation (6). This leads to a deterministic outcome where the probability q equals 1. However, there's a critical aspect to consider: the transition from a spurious solution is contingent upon the condition described in (16). A significant challenge arises if $\|M^*\|_{\bar{\Omega},F}^2$ is large, potentially rendering this bound vacuous. Here, the utility of Theorem 3.2 becomes evident. Under its probabilistic framework, we apply a triangle inequality to reduce the bound e_1 to $\sqrt{\frac{1-p+\chi}{p-\chi}}\epsilon\|M^*\|_F$. This adjustment is particularly valuable, as the presence of ϵ and p can significantly diminish the error term, effectively countering the inaccuracies introduced by our approximation method. Also note that although the bound (16) contains the term $\text{tr}(M^*)$, which we previously said was unaccessible (especially in the incoherence calculation), knowing it or not in advance will not affect whether the problem could be solved using the lifted framework, and it only affects the theoretical guarantees describing worst-case scenarios. The proof to this theorem can be found in Appendix B.

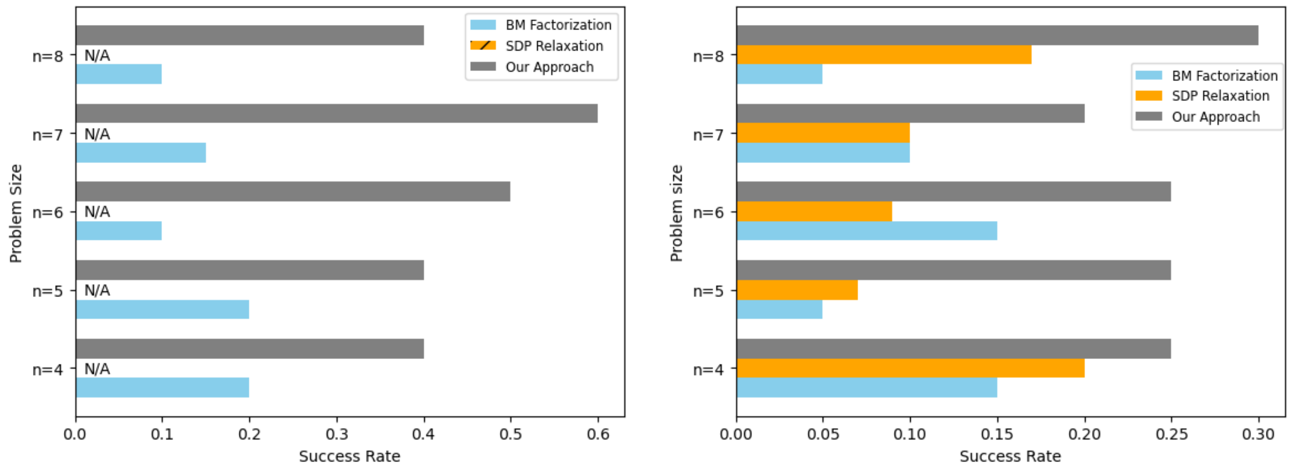
This theorem presents a new approach to matrix completion that is not reliant on the incoherence parameter or strict observation modes, diverging from established results like those in (Ge et al., 2017; Candes and Plan, 2010), which require a high sample rate on the magnitude of $\mathcal{O}(\mu_0 n^{1.2} r \log(n))$ with unknown constant scale. Our method offers a flexible tradeoff between observation probability and solution accuracy, effectively managing a gradual degradation in assurance. Furthermore, with the introduction of ϵ, it enables us to actively trade-off solution accuracy with computational complexity. This adjustment is particularly effective in noisy scenarios where some degree of inaccuracy is unavoidable, making our approach both practical and justifiable for real-world applications.

The results of Theorem 5.1 only apply to rank-1 critical points. To adhere to this, we can start our gradient descent algorithm at a small scale, leveraging Theorem A.8 to maintain the rank constraint. Theorem A.9 in Appendix A summarizes the results, and it is not presented here as its complex details could distract from the central narrative of Theorem 5.1.

6. Numerical Experiments

In this section, we numerically demonstrate the effectiveness of our method[1] against traditional BM based and semi-definite relaxation (SDP) methods.

[1]'https://github.com/anonpapersbm/mc_noisy_ms',run on M1 Max Macbook Pro



(a) Problem Instance (20)

(b) Problem Instance with Independent Entries

Figure 2. Success Rates for Our Approach compared to Standard Approaches.

To further the investigation, we introduce a benchmark matrix completion problem described in (Yalçın et al., 2022), known to be difficult:

$$\Omega = \{(i,i),(i,2k),(2k,i)| \ \forall i \in [n], k \in [\lfloor n/2 \rfloor]\}, \quad (20)$$

M^* is also chosen identically to Example 1 from (Yalçın et al., 2022) to ensure consistency. The study applies the BM factorized formulation (2), and our approach to address (20). Our approach employs the lifted problem (12) with $l = 3$, culminating in a tensor $\mathbf{w}_T$ after T iterations of gradient descent. A tensor PCA algorithm (Ma et al., 2024) extracts the principal component $X_T \in \mathbb{R}^{n \times r}$, approximating $\mathbf{w}_T$ as $\text{vec}(X_T)^{\otimes l}$. X_T then serves as the solution to the original problem (2). A successful instance of gradient descent is defined by $\|X_T X_T^\top - M^*\|_F \leq 0.05$. Preferring the success rate metric over average reconstruction error minimizes the impact of outliers and reduces variance, providing a more reliable measure of efficacy. This approach is also tested against a standard model where each entry of M^* is observed with a probability $p = 0.15$, since this was the main focus of classic matrix completion literature. Results are documented in Figure 2.

Figure 2 clearly demonstrates the superior success rate of the proposed method compared to both the BM formulation and SDP relaxation (Candès and Tao, 2010) across different settings and problem sizes n. It is noteworthy that the SDP relaxation, being a convex problem, is executed only once per scenario, as it reliably converges to a global solution. However, for the specific instance (20), the SDP approach will be invalid since there are other SDP matrices N such that $N_\Omega = M^*_\Omega$. For the independent observation model, the variability of Ω necessitates running 10 distinct problem instances for each size n, with each instance undergoing 20 trials to estimate the success rate. This testing approach showcases the higher success rate of the proposed method. Additionally, the convex relaxation typically surpasses the BM formulation, as indicated in Figure 2b. Experimental conditions included a $\epsilon = 5 * 10^{-5}$, a learning rate of 2e-2, an initialization scale of $\xi = 10^{-4}$ for (12), and the utilization of the Adam optimizer (Kingma and Ba, 2014) for all experiments except those involving the semi-definite problem, where the open-source SCS solver was employed.

7. Conclusion

In conclusion, our study introduces an approach to addressing the inherent complexities of non-convex optimization problems by adding more noise. By challenging the conventional strategy of minimizing noise to solve complex problems, our research introduces a controlled noise mechanism that not only elevates theoretical promises but also enables a strategic management of trade-offs in problem-solving. The developed ϵ-MC framework enhances practical application by allowing the integration of matrix sensing techniques, providing a flexible framework that could benefit general matrix completion problems. Our findings encourage continued exploration in actively incorporating noise and randomness into machine learning problems in order to reduce training complexity.

References

Cai, T. T. and Zhang, A. (2013). Sharp rip bound for sparse signal and low-rank matrix recovery. *Applied and Computational Harmonic Analysis*, 35(1):74–93.

Candes, E. J. and Plan, Y. (2010). Tight oracle bounds for low-rank matrix recovery from a minimal number of random measurements. *arXiv preprint arXiv:1001.0339*.

Candès, E. J. and Recht, B. (2009). Exact matrix completion via convex optimization. *Foundations of Computational Mathematics*, 9(6):717–772.

Candès, E. J. and Tao, T. (2010). The power of convex relaxation: Near-optimal matrix completion. *IEEE Transactions on Information Theory*, 56(5):2053–2080.

Comon, P., Golub, G., Lim, L.-H., and Mourrain, B. (2008). Symmetric tensors and symmetric tensor rank. *SIAM Journal on Matrix Analysis and Applications*, 30(3):1254–1279.

Dai, W. and Milenkovic, O. (2010). Set: An algorithm for consistent matrix completion. In *2010 IEEE International Conference on Acoustics, Speech and Signal Processing*, pages 3646–3649. IEEE.

Ding, L. and Chen, Y. (2020). Leave-one-out approach for matrix completion: Primal and dual analysis. *IEEE Transactions on Information Theory*, 66(11):7274–7301.

Du, S. S., Jin, C., Lee, J. D., Jordan, M. I., Singh, A., and Poczos, B. (2017). Gradient descent can take exponential time to escape saddle points. *Advances in neural information processing systems*, 30.

Fattahi, S. and Sojoudi, S. (2020). Exact guarantees on the absence of spurious local minima for non-negative rank-1 robust principal component analysis. *Journal of Machine Learning Research*, 21:1–51.

Fotopoulos, G. B., Popovich, P., and Papadopoulos, N. H. (2024). Review non-convex optimization method for machine learning. *arXiv preprint arXiv:2410.02017*.

Ge, R., Jin, C., and Zheng, Y. (2017). No spurious local minima in nonconvex low rank problems: A unified geometric analysis. In *Proceedings of the 34th International Conference on Machine Learning*, volume 70 of *Proceedings of Machine Learning Research*, pages 1233–1242.

Ge, R., Lee, J. D., and Ma, T. (2016). Matrix completion has no spurious local minimum. *Advances in neural information processing systems*, 29.

Ghassemi, M., Sarwate, A., and Goela, N. (2018). Global optimality in inductive matrix completion. In *2018 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pages 2226–2230. IEEE.

Gross, D. (2011). Recovering low-rank matrices from few coefficients in any basis. *IEEE Transactions on Information Theory*, 57(3):1548–1566.

Gu, S., Zhang, L., Zuo, W., and Feng, X. (2014). Weighted nuclear norm minimization with application to image denoising. In *Proceedings of the IEEE conference on computer vision and pattern recognition*, pages 2862–2869.

Ha, W., Liu, H., and Barber, R. F. (2020). An equivalence between critical points for rank constraints versus low-rank factorizations. *SIAM Journal on Optimization*, 30(4):2927–2955.

Haldar, J. P. and Hernando, D. (2009). Rank-constrained solutions to linear matrix equations using powerfactorization. *IEEE Signal Processing Letters*, 16(7):584–587.

Hoeffding, W. (1994). Probability inequalities for sums of bounded random variables. *The collected works of Wassily Hoeffding*, pages 409–426.

Jain, P. and Kar, P. (2017). Non-convex optimization for machine learning. *arXiv preprint arXiv:1712.07897*.

Jin, M., Molybog, I., Mohammadi-Ghazi, R., and Lavaei, J. (2019). Towards robust and scalable power system state estimation. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, pages 3245–3252. IEEE.

Keshavan, R. H., Montanari, A., and Oh, S. (2010). Matrix completion from a few entries. *IEEE transactions on information theory*, 56(6):2980–2998.

Kingma, D. P. and Ba, J. (2014). Adam: A method for stochastic optimization. *arXiv preprint arXiv:1412.6980*.

Klopp, O. (2015). Matrix completion by singular value thresholding: sharp bounds. *Electronic Journal of Statistics*, 9(2):2348–2369.

Kolda, T. G. (2015). Numerical optimization for symmetric tensor decomposition. *Mathematical Programming*, 151(1):225–248.

Koren, Y., Bell, R., and Volinsky, C. (2009). Matrix factorization techniques for recommender systems. *Computer*, 42(8):30–37.

Lasserre, J. B. (2001). Global optimization with polynomials and the problem of moments. *SIAM Journal on optimization*, 11(3):796–817.

Lee, K. and Bresler, Y. (2010). Admira: Atomic decomposition for minimum rank approximation. *IEEE Transactions on Information Theory*, 56(9):4402–4416.

Li, Q., Zhu, Z., and Tang, G. (2019). The non-convex geometry of low-rank matrix optimization. *Information and Inference: A Journal of the IMA*, 8(1):51–96.

Li, Y., Ma, T., and Zhang, H. (2018). Algorithmic regularization in over-parameterized matrix sensing and neural networks with quadratic activations. In *Conference On Learning Theory*, pages 2–47. PMLR.

Lustig, M., Donoho, D. L., Santos, J. M., and Pauly, J. M. (2008). Compressed sensing mri. *IEEE signal processing magazine*, 25(2):72–82.

Ma, J. and Fattahi, S. (2022). Global convergence of subgradient method for robust matrix recovery: Small initialization, noisy measurements, and over-parameterization. *arXiv preprint arXiv:2202.08788*.

Ma, J. and Fattahi, S. (2023). On the optimization landscape of burer-monteiro factorization: When do global solutions correspond to ground truth? *arXiv preprint arXiv:2302.10963*.

Ma, Z., Bi, Y., Lavaei, J., and Sojoudi, S. (2022). Sharp restricted isometry property bounds for low-rank matrix recovery problems with corrupted measurements. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 36, pages 7672–7681.

Ma, Z., Lavaei, J., and Sojoudi, S. (2024). Algorithmic regularization in tensor optimization: Towards a lifted approach in matrix sensing. *Advances in Neural Information Processing Systems*, 36.

Ma, Z., Molybog, I., Lavaei, J., and Sojoudi, S. (2023). Over-parametrization via lifting for low-rank matrix sensing: Conversion of spurious solutions to strict saddle points. In *International Conference on Machine Learning*. PMLR.

Mishra, B., Meyer, G., Bonnabel, S., and Sepulchre, R. (2014). Fixed-rank matrix factorizations and riemannian low-rank optimization. *Computational Statistics*, 29:591–621.

Molybog, I., Madani, R., and Lavaei, J. (2020). Conic optimization for quadratic regression under sparse noise. *The Journal of Machine Learning Research*, 21(1):7994–8029.

Nguyen, L. T., Kim, J., and Shim, B. (2019). Low-rank matrix completion: A contemporary survey. *IEEE Access*, 7:94215–94237.

Parrilo, P. A. (2003). Semidefinite programming relaxations for semialgebraic problems. *Mathematical programming*, 96(2):293–320.

Petersen, K. B., Pedersen, M. S., et al. (2008). The matrix cookbook. *Technical University of Denmark*, 7(15):510.

Recht, B. (2011). A simpler approach to matrix completion. *Journal of Machine Learning Research*, 12(12).

Recht, B., Fazel, M., and Parrilo, P. A. (2010). Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization. *SIAM Review*, 52(3):471–501.

Sun, R. and Luo, Z.-Q. (2019). Optimization for deep learning: Theory and algorithms. *SIAM Review*, 61(4):619–645.

Tanner, J. and Wei, K. (2016). Low rank matrix completion by alternating steepest descent methods. *Applied and Computational Harmonic Analysis*, 40(2):417–429.

Wang, Z., Lai, M.-J., Lu, Z., Fan, W., Davulcu, H., and Ye, J. (2014). Rank-one matrix pursuit for matrix completion. In *International Conference on Machine Learning*, pages 91–99. PMLR.

Wen, Z., Yin, W., and Zhang, Y. (2012). Solving a low-rank factorization model for matrix completion by a nonlinear successive over-relaxation algorithm. *Mathematical Programming Computation*, 4(4):333–361.

Yalcin, B., Ma, Z., Lavaei, J., and Sojoudi, S. (2023). Semidefinite programming versus burer-monteiro factorization for matrix sensing. In *Proceedings of the AAAI Conference on Artificial Intelligence*.

Yalçın, B., Zhang, H., Lavaei, J., and Sojoudi, S. (2022). Factorization approach for low-complexity matrix completion problems: Exponential number of spurious solutions and failure of gradient methods. In *International Conference on Artificial Intelligence and Statistics*, pages 319–341. PMLR.

Zhang, H., Bi, Y., and Lavaei, J. (2021). General low-rank matrix optimization: Geometric analysis and sharper bounds. *Advances in Neural Information Processing Systems*, 34:27369–27380.

Zhang, H., Yalcin, B., Lavaei, J., and Sojoudi, S. (2023). A new complexity metric for nonconvex rank-one generalized matrix completion. *Mathematical Programming*, pages 1–42.

Zhang, R. Y. (2021). Sharp global guarantees for nonconvex low-rank matrix recovery in the overparameterized regime. *arXiv preprint arXiv:2104.10790*.

Zhang, R. Y. (2022). Improved global guarantees for the nonconvex burer–monteiro factorization via rank overparameterization. *arXiv preprint arXiv:2207.01789*.

Zhang, Y., Madani, R., and Lavaei, J. (2017). Conic relaxations for power system state estimation with line measurements. *IEEE Transactions on Control of Network Systems*, 5(3):1193–1205.

Zilber, P. and Nadler, B. (2022). Inductive matrix completion: No bad local minima and a fast algorithm. In *International Conference on Machine Learning*, pages 27671–27692. PMLR.

A. Additional Details for Noisy Lifted Framework

A.1. Additional Definitions

Definition A.1 (Tensor)**.** As a generalization of the way vectors are used to parametrize finite-dimensional vector spaces, we use *arrays* to parametrize tensors generated from product of finite-dimensional vector spaces, as per (Comon et al., 2008). In particular, we define an l-way array as such:

$$\mathbf{a} = \{a_{i_1 i_2 \dots i_l} | 1 \le i_k \le n_k, 1 \le k \le l\} \in \mathbb{R}^{n_1 \times \dots \times n_l}$$

Note that in this paper tensors and arrays can be regarded as synonymous since there exists an isomorphism between them. Moreover, if $n_1 = \dots = n_l$, then we call this tensor(array) an l-order(way), n-dimensional tensor. For the convenience of tensor representation, we use the notation $\mathbb{R}^{n \circ l}$ with $n \circ l := n \times \dots \times n$. In this work, tensors are denoted with bold variables, and other fonts are reserved for matrices, vectors, and scalars unless specified otherwise.

Definition A.2 (Symmetric Tensor)**.** Similar to the definition of symmetric matrices, for an order-l tensor $\mathbf{a}$ with the same dimensions (i.e., $n_1 = \dots = n_l$), also called a cubic tensor, it is said that the tensor is symmetric if its entries are invariance under any permutation of their indices:

$$a_{i_{\sigma(1)} \cdots i_{\sigma(l)}} = a_{i_1 \cdots i_l} \quad \forall \sigma, \quad i_1, \dots, i_l \in \{1, \dots, n\}$$

where $\sigma \in \mathcal{G}_l$ denotes a specific permutation and $\mathcal{G}_l$ is the symmetric group of permutations on $\{1, \dots, l\}$. We denote the set of symmetric tensors as $\mathrm{S}^l(\mathbb{R}^n)$.

Definition A.3 (Rank of Tensors)**.** The rank of a cubic tensor $\mathbf{a} \in \mathbb{R}^{n \circ l}$ is defined as

$$\operatorname{rank}(\mathbf{a}) = \min\{r | \mathbf{a} = \sum_{i=1}^{r} u_i \otimes v_i \otimes \dots \otimes w_i\}$$

for some vector $u_i, \dots, w_i \in \mathbb{R}^n$. Furthermore, according to (Kolda, 2015), if $\mathbf{a}$ is a symmetric tensor, then it can be decomposed as:

$$\mathbf{a} = \sum_{i=1}^{r} \lambda_i u_i \otimes \dots \otimes u_i := \sum_{i=1}^{r} \lambda_i u_i^{\otimes l}$$

and the rank is conveniently defined as the number of nonzero λ_i's, which is very similar to the rank of symmetric matrices indeed. The most important concept in our paper is rank-1 tensors, and for any tensor $\mathbf{a}$, a necessary and sufficient condition for it to be rank-1 is that

$$\mathbf{a} = u^{\otimes l}$$

for some $u \in \mathbb{R}^n$.

Definition A.4 (Tensor Multiplication)**.** Outer product is an operation carried out on a pair of tensors, denoted as $\otimes$. The outer product of 2 tensors $\mathbf{a}$ and $\mathbf{b}$, respectively of orders l and p, is a tensor of order $l + p$, denoted as $\mathbf{c} = \mathbf{a} \otimes \mathbf{b}$ such that:

$$c_{i_1 \dots i_l j_1 \dots j_p} = a_{i_1 \dots i_l} b_{j_1 \dots j_p}$$

When the 2 tensors are of the same dimension, this product is such that $\otimes : \mathbb{R}^{n \circ l} \times \mathbb{R}^{n \circ p} \mapsto \mathbb{R}^{n \circ (l+p)}$. Henceforth, we use the shorthand notation

$$\underbrace{a \otimes \dots \otimes a}_{l \text{ times}} := a^{\otimes l}$$

We also define an inner product of two tensors. The mode-q inner product between the 2 aforementioned tensors having the same q-th dimension is denoted as $\langle \mathbf{a}, \mathbf{b} \rangle_q$. Without loss of generality, assume that $q = 1$ and

$$[\langle \mathbf{a}, \mathbf{b} \rangle_q]_{i_2 \dots i_l j_2 \dots j_p} = \sum_{\alpha=1}^{n_q} a_{\alpha i_2 \dots i_l} b_{\alpha j_2 \dots j_p}$$

Note that when we write $\langle \cdot, \cdot \rangle_q$, we count the q-th dimension of the first entry. Indeed, this definition of inner product can also be trivially extended to multi-mode inner products by just summing over all modes, denoted as $\langle \mathbf{a}, \mathbf{b} \rangle_{q, \dots, s}$.

Lemma A.5 (Section 10.2 (Petersen et al., 2008)). *For four arbitrary matrices A, B, C, D of compatible dimensions, it holds that*

$$\langle A \otimes B, C \otimes D \rangle_{2,4} = AC \otimes BD \tag{21}$$

Definition A.6 (Variational Eigenvalue of Tensors (Ma et al., 2024)). For a given tensor $\mathbf{w} \in \mathbb{R}^{n \circ l}$, we define its k^{th} variational eigenvalue (v-Eigenvalue) $\lambda_k^v(\mathbf{w})$ as

$$\lambda_k^v(\mathbf{w}) := \max_{\substack{S \\ \dim(S)=k}} \min_{\mathbf{u} \in S} \frac{|\langle \mathbf{w}, \mathbf{u} \rangle|}{\|\mathbf{u}\|_F^2}, \quad k \in [n]$$

where S is a subspace of $\mathbb{R}^{n \circ l}$ that is spanned by a set of orthogonal, symmetric, rank-1 tensors. Its dimension denotes the number of orthogonal tensors that span this space.

A.2. Formulation Details

As noted in our main formulation (12), the decision variable $\mathbf{w}$ is a tensor of dimension $nr \times \cdots \times nr$, since it serves as a repeated outer product of $\mathrm{vec}(X)$ with $X \in \mathbb{R}^{n \times r}$ being our original decision variable in (1) (here we assume $r_{\text{search}} = r$). The permutation $\mathbf{P}$ is needed in order to convert $\mathbf{w} \in \mathbb{R}^{nr \circ l}$ back to $\mathbb{R}^{[n \times r] \circ l}$ in order to do meaningful inner products. $\mathbf{P} \in \mathbb{R}^{n \times r \times nr}$ is defined as

$$\langle \mathbf{P}, \mathrm{vec}(X) \rangle_3 = X \quad \forall X \in \mathbb{R}^{n \times r}, n, r \in \mathbb{Z}^+$$

Such $\mathbf{P}$ can be easily constructed via filling appropriate scalar "1"s in the tensor. Via Lemma A.1, we also know that

$$\langle \mathbf{P}^{\otimes l}, \mathrm{vec}(X)^{\otimes l} \rangle_{3*[l]} = (\langle \mathbf{P}, \mathrm{vec}(X) \rangle_3)^{\otimes l} = X^{\otimes l} \tag{22}$$

Notationally, we abbreviate $\langle \mathbf{P}^{\otimes l}, \mathbf{w} \rangle_{3*[l]}$ as $\mathbf{P}(\mathbf{w})$ for enhanced readability for an arbitrary tensor $\mathbf{w}$ with dimension greater or equal to 2.

Since we also make extensive use of first and second order critical points of (12), we present them here for accessibility:

Lemma A.7. *The tensor $\hat{\mathbf{w}} \in \mathbb{R}^{nr \circ l}$ is an SOP of (12) if and only if*

$$\langle \nabla f^l(\langle \mathbf{P}(\hat{\mathbf{w}}), \mathbf{P}(\hat{\mathbf{w}}) \rangle_{2*[l]}), \mathbf{P}(\hat{\mathbf{w}}) \rangle_{2*[l]} = 0, \tag{23a}$$

$$\begin{aligned} &2\langle \nabla f^l(\langle \mathbf{P}(\hat{\mathbf{w}}), \mathbf{P}(\hat{\mathbf{w}}) \rangle_{2*[l]}), \langle \mathbf{P}(\Delta), \mathbf{P}(\Delta) \rangle_{2*[l]} + \\ &\|\langle \mathbf{A}^{\otimes l}, \langle \mathbf{P}(\hat{\mathbf{w}}), \mathbf{P}(\Delta) \rangle_{2*[l]} + \langle \mathbf{P}(\Delta), \mathbf{P}(\hat{\mathbf{w}}) \rangle_{2*[l]} \rangle\|_F^2 \geq 0 \quad \forall \Delta \in \mathbb{R}^{nr \circ l} \end{aligned} \tag{23b}$$

with (23b) being a necessary and sufficient condition for $\hat{\mathbf{w}}$ to be a FOP and $\nabla f_w^l(\mathbf{M})$ is defined as

$$\nabla f_w^l(\mathbf{M}) = \langle (\mathbf{A}^{\otimes l})^* \mathbf{A}^{\otimes l}, \mathbf{M} \rangle - [\langle \mathbf{A}^* \mathbf{A}, M^* \rangle + \langle \mathbf{A}^*, \tilde{w} \rangle]^{\otimes l} \tag{24}$$

The proof to this lemma is highly technical and can be obtained by slightly changing the proof to Lemma 7 in (Ma et al., 2024) by changing $b = \mathcal{A}(M^*)$ to $\tilde{b} = \mathcal{A}(M^*) + \tilde{w}$ defined above.

Here we present the full theorem of Theorem 4.2 regarding implicit bias in (12):

Theorem A.8. *Consider a finite-horizon gradient descent trajectory $\{\mathbf{w}_t\}_{t \in [T]}$ of (12) with $\mathbf{w}_{t+1} = \mathbf{w}_t - \eta \nabla h^l(\mathbf{w}_t)$ starting from the initialization $\mathbf{w}_0 = \xi x_0^{\otimes l}$ with $\xi \in \mathbb{R}$ denoting the scale of the initialization, η representing the step-size and $x_0 \in \mathbb{R}^{nr}$ being an arbitrary vector with $\|x_0\|_2^2 = 1$. Then there exists $t(\kappa, l) \geq 1$ and $\kappa < 1$ such that*

$$\frac{\lambda_2^v(\mathbf{w}_t)}{\lambda_1^v(\mathbf{w}_t)} \leq \kappa, \qquad \forall t \in [t(\kappa, l), T] \tag{25}$$

if the initialization scale ξ is sufficiently small, where $t(\kappa, l)$ is expressed as

$$t(\kappa, l) = \left\lceil \ln\left(\frac{\|x_0\|_2^l}{\kappa |v_1^\top x_0|^l}\right) \ln\left(\frac{1 + \eta\sigma_1^l(U)}{1 + \eta\sigma_2^l(U)}\right)^{-1} \right\rceil \tag{26}$$

where $\sigma_1(U)$ and $\sigma_2(U)$ denote the first and second singular values of U and v_1, v_2 are the associated singular vectors, with

$$U = \langle \mathbf{A}_r, \tilde{b} \rangle_1 \in \mathbb{R}^{nr \times nr}, \quad \mathbf{A}_r := I_r \oslash_{2,3} \mathbf{A} \tag{27}$$

Next, we present a technical extension of Theorem 5.1 and Theorem A.8, showing how gradient descent initialized with small scale can help ensure that second-order points of lifted version of (7) remain very close to M^* along the optimization trajectory

Theorem A.9. *Consider a generic matrix completion problem under the same premise as given in Theorem 5.1. Assume that the symmetric tensor $\hat{\mathbf{w}} \in \mathbb{R}^{nr \circ l}$ is a second-order point (local minima) of (12) that is κ-rank-1 with $\kappa \leq \mathcal{O}(1/\|M^*\|_F^2)$. This can be achieved by initializing the vanilla gradient algorithm at $\mathbf{w}_0 = \xi x_0^{\otimes l}$ with a sufficiently small $\xi > 0 \in \mathbb{R}$. Then after iterations $t(\kappa, l)$ given in (26), Theorem A.8 ensures that all tensors along the trajectory will become κ-rank-1.*

If $\hat{\mathbf{w}}$'s major spectral decomposition is given as $\hat{\mathbf{w}} = \lambda_S \hat{x}^{\otimes l} + \hat{\mathbf{w}}^\dagger$ with $\hat{x} \in \mathbb{R}^{nr}$ being a FOP of (7) (ensured by Proposition 2 in (Ma et al., 2024)), we know that

$$\|M^* - \hat{X}\hat{X}^\top\|_F < \frac{1}{\epsilon}\lambda_r(\hat{X})\sqrt{\operatorname{tr}(M^*)} + \mathcal{O}(\sqrt{r}\kappa^{1/2l}) + e_1 \tag{28}$$

holds with probability at least q, under the condition that l is odd and meets the requirement:

$$l > \frac{1}{1 - \log_2(2\beta)}, \quad \beta := \frac{\operatorname{tr}(M^*)\lambda_r(\hat{X}\hat{X}^\top)}{\epsilon^2\left(\|M^* - \hat{X}\hat{X}^\top\|_F^2 - \mathcal{O}(r\kappa^{1/l}) - e_2\right)}. \tag{29}$$

where e_1, e_2, and q are identical to those given in Theorem 5.1 depending on different MC instances.

The proof of this Theorem is omitted because it directly follows from Theorem 5.1, Theorem A.8, and Theorem 2 in (Ma et al., 2024).

B. Missing Proofs

Proof to Theorem 3.2. To begin with, we reiterate our elementary results which follows from the definition of $M^\dagger$ and that of (7):

$$\begin{aligned}
&0 \geq f_{w_\epsilon}(M^\dagger) - f_{w_\epsilon}(M^*)\\
=&\frac{1}{2}\|\mathcal{A}_{\Omega,\epsilon}(M^\dagger - M^*)\|_{\bar{S},2}^2 + \frac{1}{2}\|\mathcal{A}_{\Omega,\epsilon}(M^\dagger - M^*) - w_\epsilon\|_{S,2}^2 - \frac{1}{2}\|w_\epsilon\|_{S,2}^2\\
=&\frac{1}{2}\|\mathcal{A}_{\Omega,\epsilon}(M^\dagger - M^*)\|_{\bar{S},2}^2 + \frac{1}{2}\|\mathcal{A}_{\Omega,\epsilon}(M^\dagger)\|_{S,2}^2 - \frac{1}{2}\epsilon^2\|M^*\|_{S,2}^2\\
\geq&\frac{1}{2}\|\mathcal{A}_{\Omega,\epsilon}(M^\dagger - M^*)\|_{\bar{S},2}^2 - \frac{1}{2}\epsilon^2\|M^*\|_{S,2}^2
\end{aligned}$$

This follows from the simple observation that

$$(w_\epsilon)_i = -\epsilon \operatorname{vec}(M^*)_i \quad \forall\ i \in [n^2]$$

Then moving $\frac{1}{2}\epsilon^2\|M^*\|_{S,2}^2$ to the left hand side, and adding $\frac{1}{2}\|M^\dagger - M^*\|_{S,2}^2$ to both sides gives

$$\|M^\dagger - M^*\|_F^2 \leq \|M^\dagger - M^*\|_S^2 + \epsilon^2\|M^*\|_{S,2}^2 \tag{30}$$

If we define a new vector $d \in \mathbb{R}^{n^2}$ in which

$$d := \operatorname{vec}(M^\dagger - M^*) \odot \operatorname{vec}(M^\dagger - M^*) + \epsilon^2 \operatorname{vec}(M^*) \odot \operatorname{vec}(M^*)$$

then we know that

$$d_i = \left(\operatorname{vec}(M^\dagger)_i - \operatorname{vec}(M^*)_i\right)^2 + \epsilon^2 \operatorname{vec}(M^*)_i^2 \geq 0 \quad \forall\ i \in [n^2]$$

So if we further define a series of random variables $\{r_1, r_2, \ldots, r_{n^2}\}$ with

$$r_i = \begin{cases} 0 & \text{with probability } p \\ d_i & \text{with probability } 1-p \end{cases} \tag{31}$$

Then we know that

$$\|M^\dagger - M^*\|_S^2 + \epsilon^2\|M^*\|_{S,2}^2 = \sum_{i=1}^{n^2} r_i := R \tag{32}$$

because for any matrix $M \in \mathbb{R}^{n_1 \times n_2}$, we have

$$\|M\|_S^2 = \sum_i^{n_1 n_2} m_i^2, \quad m_i = \begin{cases} 0 \quad \text{with probability } p \\ \mathrm{vec}(M)_i \quad \text{with probability } 1-p \end{cases}$$

Then we simply acknowledge that $0 \le r_i \le d_i$ almost surely, which sets up the premise to use Hoeffding's inequality (Hoeffding, 1994). This concentration inequality gives that

$$\mathbb{P}\left(R \le \mathbb{E}[R] + t\right) \ge 1 - \exp\left(\frac{-2t^2}{\sum_i^{n^2}(d_i - 0)^2}\right) = 1 - \exp\left(\frac{-2t^2}{\|d\|_2^2}\right) \tag{33}$$

First of all, we could easily derive that

$$\begin{aligned} \mathbb{E}[R] &= \sum_i^{n^2}(1-p)\left[\left(\mathrm{vec}(M^\dagger)_i - \mathrm{vec}(M^*)_i\right)^2 + \epsilon^2 \,\mathrm{vec}(M^*)_i^2\right] \\ &= (1-p)\left(\|M^\dagger - M^*\|_F^2 + \epsilon^2\|M^*\|_F^2\right) \end{aligned} \tag{34}$$

Therefore combining (30), (33) and (34) we have

$$\mathbb{P}\left(\|M^\dagger - M^*\|_F^2 \le (1-p)\left(\|M^\dagger - M^*\|_F^2 + \epsilon^2\|M^*\|_F^2\right) + t\right) \ge 1 - \exp\left(\frac{-2t^2}{\|d\|_2^2}\right) \tag{35}$$

Then we can choose

$$t = \chi\left(\|M^\dagger - M^*\|_F^2 + \epsilon^2\|M^*\|_F^2\right) = \chi\|d\|_1$$

for some constant $\chi \le p$. This will then transform (35) into

$$\begin{aligned} &\mathbb{P}\left(\|M^\dagger - M^*\|_F^2 \le (1-p+\chi)\left(\|M^\dagger - M^*\|_F^2 + \epsilon^2\|M^*\|_F^2\right)\right) \ge 1 - \exp\left(\frac{-2t^2}{\|d\|_2^2}\right) \\ \implies &\mathbb{P}\left((p-\chi)\|M^\dagger - M^*\|_F^2 \le (1-p+\chi)\epsilon^2\|M^*\|_F^2\right) \ge 1 - \exp\left(\frac{-2\chi^2\|d\|_1^2}{\|d\|_2^2}\right) \\ \implies &\mathbb{P}\left(\|M^\dagger - M^*\|_F^2 \le \frac{1-p+\chi}{p-\chi}\epsilon^2\|M^*\|_F^2\right) \ge 1 - \exp\left(\frac{-2\chi^2\|d\|_1^2}{\|d\|_2^2}\right) \end{aligned} \tag{36}$$

which proves our desired result directly. □

Proof of Theorem 4.1. First of all, we hope to decompose the Hessian of (1) at a second order point $\hat{X} \in \mathbb{R}^{n \times r}$. Classic matrix sensing literatures like (Ha et al., 2020; Zhang et al., 2021; Li et al., 2019) give that the second-order critical condition of (1) are given as

$$\nabla f(\hat{X}\hat{X}^\top)\hat{X} = 0, \tag{37}$$

$$2\langle \nabla f(\hat{X}\hat{X}^\top), UU^\top\rangle + [\nabla^2 f(\hat{X}\hat{X}^\top)](\hat{X}U^\top + U\hat{X}^\top, \hat{X}U^\top + U\hat{X}^\top) \ge 0 \quad \forall U \in \mathbb{R}^{n\times r} \tag{38}$$

with (37) being the first order critical condition. Moreover, since the sensing matrices $\{A_i\}_{i\in[m]}$ can be assumed be to symmetric without loss of generality (Zhang et al., 2021), we have that

$$[\nabla^2 f(\hat{X}\hat{X}^\top)](\hat{X}U^\top + U\hat{X}^\top, \hat{X}U^\top + U\hat{X}^\top) = 4[\nabla^2 f(\hat{X}\hat{X}^\top)](\hat{X}U^\top, \hat{X}U^\top).$$

We then could decompose LHS of (38) as $2C_1 + 4C_2$ where

$$C_1 \coloneqq \langle \nabla f(\hat{X}\hat{X}^\top), UU^\top\rangle, \quad C_2 \coloneqq [\nabla^2 f(\hat{X}\hat{X}^\top)](\hat{X}U^\top, \hat{X}U^\top)$$

Given the assumption that (1) obeys some RSS condition, it is possible to upper-bound C_2 by observing

$$[\nabla^2 f(\hat{X}\hat{X}^\top)](\hat{X}U^\top + U\hat{X}^\top, \hat{X}U^\top + U\hat{X}^\top) \le L_s\|\hat{X}U^\top + U\hat{X}^\top\|_F^2$$

Therefore, if want to somehow create an negative escape direction for $\hat{X}$, it is important that we find a U such that C_1 is negative and large in magnitude, and then amplify this term via tensor parametrization. To do so, we first do a more in-depth analysis of $\nabla f(\hat{X}\hat{X}^\top)$. As mentioned above, since $\nabla f(\cdot)$ can be assumed to be symmetric, one can select $u \in \mathbb{R}^n$ such that $u^\top \nabla f(\hat{x}\hat{x}^\top)u = \lambda_{\min}(\nabla f(\hat{x}\hat{x}^\top))$. Then via the definition of RSC we have

$$f(M^*) \geq f(\hat{X}\hat{X}^\top) + \langle \nabla f(\hat{X}\hat{X}^\top), M^* - \hat{X}\hat{X}^\top \rangle + \frac{\alpha_s}{2}\|\hat{X}\hat{X}^\top - M^*\|_F^2. \tag{39}$$

With $\hat{X}$ being a first-order point, according to (37)

$$\nabla f(\hat{X}\hat{X}^\top)\hat{X} = 0 \implies \langle \nabla f(\hat{X}\hat{X}^\top), \hat{X}\hat{X}^\top \rangle = 0$$

Therefore, if in (1) our b is corrupted as $\mathcal{A}(M^*) + \tilde{w}$, then plugging it back into (39) gives

$$\begin{aligned} \langle \nabla f(\hat{X}\hat{X}^\top), M^* \rangle &\leq -\frac{\alpha_s}{2}\|\hat{x}\hat{x}^\top - M^*\|_F^2 + f(M^*) - f(XX^\top) \\ &\leq -\frac{\alpha_s}{2}\|\hat{x}\hat{x}^\top - M^*\|_F^2 + f(M^*) \\ &= -\frac{\alpha_s}{2}\|\hat{x}\hat{x}^\top - M^*\|_F^2 + \frac{\|\tilde{w}\|_2^2}{2} \end{aligned} \tag{40}$$

where the second inequality follows from the fact that $f(\cdot) \geq 0$ in its entire domain and the last inequality follows from $f(M^*) = 1/2\|\mathcal{A}(M^*) - \mathcal{A}(M^*) - \tilde{w}\|_2^2 = \|\tilde{w}\|_2^2/2$. Furthermore, since both $\nabla f(\hat{X}\hat{X}^\top)$ and M^* are assumed to be positive semidefinite,

$$\langle \nabla f(\hat{X}\hat{X}^\top), M^* \rangle \geq \lambda_{\min}(\nabla f(\hat{X}\hat{X}^\top)) \operatorname{tr}(M^*)$$

which implies that

$$\lambda_{\min}(\nabla f(\hat{X}\hat{X}^\top)) \leq \frac{-\alpha_s\|\hat{X}\hat{X}^\top - M^*\|_F^2 + \|\tilde{w}\|_2^2}{2\operatorname{tr}(M^*)} \tag{41}$$

Furthermore, (13) gives us

$$\|\hat{X}\hat{X}^\top - M^*\|_F^2 \geq \|\tilde{w}\|_2^2/\alpha_s$$

since $\frac{L_s}{\alpha_s}\lambda_r(\hat{X}\hat{X}^\top)\operatorname{tr}(M^*) \geq 0$ by definition. This means that

$$\lambda_{\min}(\nabla f(\hat{X}\hat{X}^\top)) \leq \frac{-\alpha_s\|\hat{X}\hat{X}^\top - M^*\|_F^2 + \|\tilde{w}\|_2^2}{2\operatorname{tr}(M^*)} \leq 0 \tag{42}$$

Thus, with this result equipped, we can further find a U that makes C_1 small. In the most convenient manner, we first consider the eigenvector $u \in \mathbb{R}^n$ of $\nabla f(\hat{X}\hat{X}^\top)$ associated with $\lambda_{\min}(\nabla f(\hat{X}\hat{X}^\top))$. Additionally we consider $q \in \mathbb{R}^r$ to be the r-th singular value of $\hat{X}$, with

$$\|\hat{X}q\|_2 = \sigma_r(\hat{X}), \qquad \|q\|_2 = 1$$

Then choosing $U \in \mathbb{R}^{n\times r} = uq^\top$ leads to

$$C_1 = \langle \nabla f(\hat{X}\hat{X}^\top), UU^\top \rangle = \langle \nabla f(\hat{X}\hat{X}^\top), uu^\top \rangle = -G$$

where $G \coloneqq -\lambda_{\min}(\nabla f(\hat{X}\hat{X}^\top)) \geq 0$. By recalling $\hat{X}^\top u = 0$ according to the first-order condition (37), we can further bound C_2 with this chocie of U as

$$\begin{aligned} L_s\|\hat{X}U^\top + U\hat{X}^\top\|_F^2 &= L_s\|u(\hat{X}q)^\top + (\hat{X}q)u^\top\|_F^2 \\ &= 2L_s\|\hat{X}q\|_F^2 + 2L_s(q^\top(\hat{X}^\top u))^2 \\ &= 2L_s\lambda_r(\hat{X}\hat{X}^\top), \end{aligned}$$

leading to

$$C_2 \leq \frac{1}{2}L_s\lambda_r(\hat{X}\hat{X}^\top)$$

Now, if we choose $\Delta = \text{vec}(U)^{\otimes l}$ for the aforementioned $U \in \mathbb{R}^{n\times r}$, the LHS of (23b) can be expressed as:

$$\begin{aligned} &2(\langle \mathbf{A}, \hat{X}\hat{X}^\top\rangle_{2,3}^\top \langle \mathbf{A}, uu^\top\rangle_{2,3})^l - 2\left((\langle \mathbf{A}, M^*\rangle_{2,3} + \tilde{w})^\top \langle \mathbf{A}, uu^\top\rangle_{2,3}\right)^l + 4(\|\langle \mathbf{A}, \hat{X}U^\top\rangle_{2,3}\|_2^2)^l \\ \le& 2(\lambda_{\min}(\nabla f(\hat{X}\hat{X}^\top)))^l + 4C_2^l \\ =& 2C_1^l + 4C_2^l \end{aligned} \tag{43}$$

where the inequality follows from:

$$a^n - b^n \le (a-b)^n, \quad \forall b \ge a \ge 0$$

Here, since $a - b = C_1 \le 0$, the above inequality can be used. As a result,

$$\text{LHS of (23b)} \le \underbrace{-2G^l}_{\text{Part 1}} + \underbrace{\frac{2}{2^{l-1}} L_s^l \lambda_r(\hat{X}\hat{X}^\top)^l}_{\text{Part 2}}$$

We know since $G \ge 0$, Part 1 is always negative assuming l is odd, and Part 2 is always positive. Therefore, it suffices to find an order l such that

$$G^l > (1/2^{l-1}) L_s^l \lambda_r(\hat{X}\hat{X}^\top)^l \tag{44}$$

Conveniently, (42) says that

$$G \ge \frac{\alpha_s \|M^* - \hat{X}\hat{X}^\top\|_F^2 - \|\tilde{w}\|_2^2}{2\,\text{tr}(M^*)}, \tag{45}$$

which can be used to derive sufficient condition for (44). Therefore, if

$$\left(\frac{\alpha_s \|M^* - \hat{X}\hat{X}^\top\|_F^2 - \|\tilde{w}\|_2^2}{2\,\text{tr}(M^*)}\right)^l > (1/2^{l-1}) L_s^l \lambda_r(\hat{X}\hat{X}^\top)^l,$$

we can conclude that (44) holds, which implies that the LHS of (23b) is negative, directly proving that $\hat{X}^{\otimes l}$ is not an SOP anymore. Elementary manipulations of the above equation give that a sufficient condition is

$$\|M^* - \hat{X}\hat{X}^\top\|_F^2 - \|\tilde{w}\|_2^2/\alpha_s > 2^{1/l}\frac{L_s}{\alpha_s}\lambda_r(\hat{X}\hat{X}^\top)\,\text{tr}(M^*) \tag{46}$$

We now consider (13), which means that

$$\lambda_r(\hat{X}\hat{X}^\top) \le \frac{\alpha_s \|M^* - \hat{X}\hat{X}^\top\|_F^2 - \|\tilde{w}\|_2^2}{L_s\,\text{tr}(M^*)} \tag{47}$$

Subsequently, define a constant γ such that

$$L_s \lambda_r(\hat{X}\hat{X}^\top) = \gamma\left[\frac{\alpha_s \|M^* - \hat{X}\hat{X}^\top\|_F^2 - \|\tilde{w}\|_2^2}{2\,\text{tr}(M^*)}\right]$$

Then, (45) and (47) together imply that $1 \le \gamma < 2$. Using this simplified notation, our sufficient condition (46) becomes

$$1 > \frac{\gamma}{2^{(l-1)/l}} \tag{48}$$

Given $1 \le \gamma < 2$, there always exists a large enough l such that (48) holds, which proves that LHS of (23b) is negative, proving that $\text{vec}(\hat{X})^{\otimes l}$ is a strict saddle, concluding the proof.

To derive a sufficient l, we simply acknowledge

$$\gamma = \frac{2L_s\,\text{tr}(M^*)\lambda_r(\hat{X}\hat{X}^\top)}{\alpha_s \|M^* - \hat{X}\hat{X}^\top\|_F^2 - \|\tilde{w}\|_2^2} := 2\beta$$

and that $\beta \le 1$ due to assumption (13). Therefore, for (48) to hold true, it is enough to have

$$2^{(l-1)/l} > 2\beta \implies \frac{l-1}{l} > \log_2(2\beta) \implies l > \frac{1}{1-\log_2(2\beta)}$$

$\square$

Proof of Theorem A.8. First of all, we hope to decompose the GD trajectory of (12) $\{\mathbf{w}_t\}_{t=0}^T$ as follows:

$$\mathbf{w}_{t+1} = \langle \mathbf{Z}_t, \mathbf{w}_0 \rangle - \mathbf{E}_t \coloneqq \tilde{\mathbf{w}}_t - \mathbf{E}_t \tag{49}$$

where

$$\begin{aligned}
\mathbf{Z}_t &\coloneqq (\mathcal{I} + \eta \langle \mathbf{A}_r^{\otimes l}, \tilde{b}^{\otimes l} \rangle)^t, \quad \mathbf{A}_r = I_r \oslash_{2,3} A \\
\mathbf{E}_t &\coloneqq \sum_{i=1}^t (\mathcal{I} + \eta \langle \mathbf{A}_r^{\otimes l}, \tilde{b}^{\otimes l} \rangle)^{t-i} \hat{\mathbf{E}}_i \\
\hat{\mathbf{E}}_i &\coloneqq \eta \langle \langle (\mathbf{A}_r^l)^* \mathbf{A}^l, \langle \mathbf{P}(\mathbf{w}_{i-1}), \mathbf{P}(\mathbf{w}_{i-1}) \rangle_{2*[l]} \rangle, \mathbf{w}_{i-1} \rangle_{2*[l]} \\
(\mathbf{A}_r^l)^* \mathbf{A}^l &\coloneqq \langle (\mathbf{A}_r)^{\otimes l}, \mathbf{A}^{\otimes l} \rangle_{3,6,\dots,3l} \in \mathbb{R}^{[nr \times nr \times n \times n] \circ l}
\end{aligned}$$

This can be proved via induction where

$$\begin{aligned}
\mathbf{w}_1 &= \left(\mathcal{I} + \eta \langle \mathbf{A}_r^{\otimes l}, \tilde{b}^{\otimes l} - (\mathbf{A}^{\otimes l})^* \langle \mathbf{P}(\mathbf{w}_0), \mathbf{P}(\mathbf{w}_0) \rangle \right) \mathbf{w}_0 \\
&= (\mathcal{I} + \eta \langle \mathbf{A}_r^{\otimes l}, \tilde{b}^{\otimes l} \rangle) \mathbf{w}_0 - \eta \langle (\mathbf{A}_r^l)^* \mathbf{A}^l, \langle \mathbf{P}(\mathbf{w}_0), \mathbf{P}(\mathbf{w}_0) \rangle \rangle \mathbf{w}_0 \\
&= \langle \mathbf{Z}_1, \mathbf{w}_0 \rangle - \mathbf{E}_1
\end{aligned}$$

This serves as our base case, and the induction step can be proven as

$$\begin{aligned}
\mathbf{w}_{t+1} &= \left(\mathcal{I} + \eta \langle \mathbf{A}_r^{\otimes l}, \tilde{b}^{\otimes l} - (\mathbf{A}^{\otimes l})^* \langle \mathbf{P}(\mathbf{w}_t), \mathbf{P}(\mathbf{w}_t) \rangle \right) \mathbf{w}_t \\
&= (\mathcal{I} + \eta \langle \mathbf{A}_r^{\otimes l}, \tilde{b}^{\otimes l} \rangle) \mathbf{w}_t - \eta \langle (\mathbf{A}_r^l)^* \mathbf{A}^l, \langle \mathbf{P}(\mathbf{w}_t), \mathbf{P}(\mathbf{w}_t) \rangle \rangle \mathbf{w}_t \\
&= (\mathcal{I} + \eta \langle \mathbf{A}_r^{\otimes l}, \tilde{b}^{\otimes l} \rangle) \mathbf{w}_t - \hat{\mathbf{E}}_{t+1} \\
&= (\mathcal{I} + \eta \langle \mathbf{A}_r^{\otimes l}, \tilde{b}^{\otimes l} \rangle) \left(\tilde{\mathbf{w}}_t - \sum_{i=1}^t (\mathcal{I} + \eta \langle \mathbf{A}_r^{\otimes l}, \tilde{b}^{\otimes l} \rangle)^{t-i} \hat{\mathbf{E}}_i \right) - \hat{\mathbf{E}}_{t+1} \\
&= \tilde{\mathbf{w}}_{t+1} - \sum_{i=1}^t (\mathcal{I} + \eta \langle \mathbf{A}_r^{\otimes l}, \tilde{b}^{\otimes l} \rangle)^{t+1-i} \hat{\mathbf{E}}_i - \hat{\mathbf{E}}_{t+1} \\
&= \tilde{\mathbf{w}}_{t+1} - \sum_{i=1}^{t+1} (\mathcal{I} + \eta \langle \mathbf{A}_r^{\otimes l}, \tilde{b}^{\otimes l} \rangle)^{t+1-i} \hat{\mathbf{E}}_i \\
&= \tilde{\mathbf{w}}_{t+1} - \mathbf{E}_t
\end{aligned}$$

Therefore, we can then use a version of Lemma 13 and Lemma 2 in (Ma et al., 2024) with $U = \langle \mathbf{A}_r^* \mathbf{A}, M^* \rangle$ replaced with $U = \langle \mathbf{A}_r, \tilde{b} \rangle_1$ to prove this theorem, following the steps in proof to Theorem 1 in (Ma et al., 2024). □

Proof to Theorem 5.1. For the general matrix completion case, as promised by Lemma 3.1, the theorem is proved by substituting $L_s = 1$, $\alpha_s = \epsilon^2$ into Theorem 5.1, and since this is a deterministic result, it happens with probability 1, meaning that for any second-order point $\hat{\mathbf{w}} = \hat{X}^{\otimes l}$ of (12), it satisfies that

$$\begin{aligned}
\|M^* - \hat{X}\hat{X}^\top\|_F &< \sqrt{\frac{L_s}{\alpha_s} \lambda_r(\hat{X}\hat{X}^\top) \operatorname{tr}(M^*) + \frac{\epsilon^2 \|M^*\|_{\bar{\Omega},F}^2}{\epsilon^2}} \\
&= \sqrt{\frac{L_s}{\alpha_s} \lambda_r(\hat{X}\hat{X}^\top) \operatorname{tr}(M^*) + \|M^*\|_{\bar{\Omega},F}^2} \\
&\le \sqrt{\frac{L_s}{\alpha_s} \lambda_r(\hat{X}\hat{X}^\top) \operatorname{tr}(M^*)} + \|M^*\|_{\bar{\Omega},F} \\
&= \frac{1}{\epsilon} \lambda_r(\hat{X}) \sqrt{\operatorname{tr}(M^*)} + \|M^*\|_{\bar{\Omega},F}
\end{aligned} \tag{50}$$

where l has to obey equation (14) as stated in Theorem 4.1.

In the case of each entry of M^* being observed independently with probability p, we first apply Theorem 5.1 with $\tilde{w} = 0$ to (7), meaning that we first assume that no noise exists in b. This is the case where we are actually trying to recover the global solution of (7), denoted as $M^\dagger$. This means that for any rank-1 critical point $\hat{\mathbf{w}} = \hat{X}^{\otimes l}$ of (12), it is a second-order point only if

$$\|\hat{X}\hat{X}^\top - M^\dagger\|_F^2 < \frac{1}{\epsilon^2}\lambda_r(\hat{X}\hat{X}^\top)\,\mathrm{tr}(M^*) \tag{51}$$

holds, when l is odd and satisfies

$$l > \frac{1}{1-\log_2(2\beta)}, \quad \beta := \frac{L_s\,\mathrm{tr}(M^*)\lambda_r(\hat{X}\hat{X}^\top)}{\epsilon^2\|M^* - \hat{X}\hat{X}^\top\|_F^2}. \tag{52}$$

The above statement holds deterministically. However, Theorem 3.2 also tells us that

$$\|M^\dagger - M^*\|_F^2 \le \frac{1-p+\chi}{p-\chi}\epsilon^2\|M^*\|_F^2$$

with high probability, so then by a triangle inequality we have that the conversion criterion above transforms to

$$\begin{aligned}\|\hat{X}\hat{X}^\top - M^*\|_F &\le \|\hat{X}\hat{X}^\top - M^\dagger\|_F + \|M^\dagger - M^*\|_F \\ &< \frac{\lambda_r(\hat{X})\sqrt{\mathrm{tr}(M^*)}}{\epsilon} + \sqrt{\frac{1-p+\chi}{p-\chi}}\epsilon\|M^*\|_F\end{aligned} \tag{53}$$

with the same probability stated in Theorem 3.2, thereby concluding the proof. □